Rational group ring elements with kernels having irrational dimension

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Abstract

We prove that there are examples of finitely generated groups Γ together with group ring elements $Q \in \mathbb{Q}\Gamma$ for which the von Neumann dimension $\dim_{L\Gamma} \ker Q$ is irrational, so (in conjunction with other known results) answering a question of Atiyah.

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1 Introduction

Given a countable discrete group Γ, we write $\mathbb{Q}\Gamma$ and $\mathbb{C}\Gamma$ respectively for its rational and complex group rings, $\lambda : \Gamma \curvearrowright \ell^2(\Gamma)$ for the Hilbertian completion of its left regular representation and $L\Gamma$ for the resulting group von Neumann algebra, which may be obtained by completing $\lambda(\mathbb{C}\Gamma)$ in the weak operator topology of $\mathcal{B}(\ell^2(\Gamma))$. Henceforth we will generally identify $\mathbb{Q}\Gamma$ and $\mathbb{C}\Gamma$ with their images in $L\Gamma$ under λ. In this setting we can define the von Neumann dimension of any closed $L\Gamma$-submodule of $\ell^2(\Gamma)$; we assume familiarity with this notion, referring the reader to the book of Lück [10] for an introduction. We will address the following classical question:

> Do there exist Γ and $Q \in \mathbb{Q}\Gamma$ for which the von Neumann dimension $\dim_{L\Gamma} \ker Q$ is irrational?

This is known to be equivalent to the problem posed by Atiyah of constructing a cocompact free proper Γ-manifold without boundary that has irrational L^2-Betti numbers (originally formulated as problem (iii) on page 72 of [1]). This equivalence is proved in Lemma 10.5 of Lück [10]: in particular, it is proved that given any Γ and $A \in \mathbb{Q}\Gamma$ one can construct a cocompact free proper Γ-manifold one of whose L^2-Betti numbers is equal to $\dim_{L\Gamma} \ker A$. We will henceforth restrict our attention to the purely group-theoretic version of the problem. A much more thorough discussion of this question is contained in Lück's [10] Chapter 10, and a discussion of its relation to questions of computability can be found in section 8.A_4 of Gromov's essay in [11].

A stronger version of the question, asking whether in fact $\dim_{L\Gamma} \ker Q$ must always lie in the additive subgroup $\mathrm{fin}^{-1}(\Gamma) \leq \mathbb{Q}$ generated by the inverses of the orders of the finite subgroups of Γ, is now known to be false from the work [8] of Grigorchuk and Żuk (see also the article [7] of Grigorchuk, Linnel, Schick and Żuk), who have shown that the lamplighter group $\mathbb{Z}_2 \wr \mathbb{Z}$ is a counterexample: all of its finite subgroups have order that is a member of $2^{\mathbb{Z}}$, but a natural finitely-supported operator with integer coefficients on the group (in fact, a rational multiple of a Markov operator) has an eigenspace with von Neumann dimension $\frac{1}{3}$.

A new and quite elementary treatment of this fact has now been given by Dicks and Schick in [5], and in this work we will adapt some of their calculations to provide a family of examples answering the original question about irrational dimensions, as formulated above. In order to state our main theorem precisely we first need a little notation.

We write $\mathbf{F}_n$ to denote the free group on n generators, s_1, s_2, $\dots$, s_n for those generators themselves, $S = \{s_1^{\pm 1}, s_2^{\pm 1}, \dots, s_n^{\pm 1}\}$ for the corresponding symmetric generating set and e for the identity element of $\mathbf{F}_n$. To these data are associated the Cayley graph $\mathrm{Cay}(\mathbf{F}_n, S)$ with vertex set $\mathbf{F}_n$ and edge set $\{\{g, gs\} : g \in \mathbf{F}_n,\ s \in S\}$, which is simply a $2n$-regular infinite tree. Here and later in the paper we will use mostly standard graph-theoretic terminology in relation to $\mathrm{Cay}(\mathbf{F}_n, S)$, as described, for instance, in Chapter I of Bollobás [3]. Given a subset $A \subset \mathbf{F}_n$ we will write $\mathrm{Cay}(\mathbf{F}_n, S)|_A$ for the induced subgraph of $\mathrm{Cay}(\mathbf{F}_n, S)$ on the set of vertices A, and

$$\partial A := A \cdot S \setminus A$$

for the **boundary** of A in $\mathrm{Cay}(\mathbf{F}_n, S)$. A **path** in $\mathrm{Cay}(\mathbf{F}_2, S)$ is a subset $P = \{g_0, g_1, \dots, g_\ell\} \subset \mathbf{F}_2$ with $g_{i+1} \in g_i S$ for every $i \leq \ell - 1$ and with all the g_is distinct, and in this case the **length** of the path is ℓ. We denote by ρ the left-invariant word metric on $\mathbf{F}_2$, which is simply the graph distance arising from $\mathrm{Cay}(\mathbf{F}_n, S)$, and will sometimes refer to $\rho(e, g)$ as the **length** of an element $g \in \mathbf{F}_n$. Given $g \in \mathbf{F}_n$ and $r \geq 0$ we let $B(g, r) := \{h \in \mathbf{F}_n : \ \rho(g, h) \leq r\}$ be the closed ball of radius r around g in $\mathrm{Cay}(\mathbf{F}_n, S)$, and more generally given $A \subseteq \mathbf{F}_n$ we let $B(A, r) := \bigcup_{g \in A} B(g, r)$ be its **radius-r neighbourhood**.

In addition we write $\mathbb{Z}_2$ to denote the cyclic group of order 2, and $\mathbb{Z}_2^{\oplus I}$ (respectively $\mathbb{Z}_2^I$) to denote the direct sum (respectively direct product) of a family of copies of $\mathbb{Z}_2$ indexed by some other set I. We will usually denote members of $\mathbb{Z}_2^{\oplus I}$ by lowercase bold letters such as $\mathbf{w} = (w_i)_{i \in I}$, and will write δ_i for the distinguished element of $\mathbb{Z}_2^{\oplus I}$ that takes the value $1 \in \mathbb{Z}_2$ at i and 0 elsewhere.

The main result of this paper is the following.

Theorem 1.1 *Let the space $\mathcal{P}(\mathbb{N})$ of subsets of $\mathbb{N}$ be endowed with the lexicographic ordering. There are parameterizations*

$$\mathcal{P}(\mathbb{N}) \ni I \mapsto V_I \leq \mathbb{Z}_2^{\oplus \mathbf{F}_2}$$

of a family of subgroups that are invariant under the left-coordinate-translation action of $\mathbf{F}_2$ and

$$I \mapsto Q_I \in \mathbb{Q}((\mathbb{Z}_2^{\oplus \mathbf{F}_2}/V_I) \rtimes \mathbf{F}_2)$$

of a family of rational group ring elements such that the associated map

$$\mathcal{P}(\mathbb{N}) \to \mathbb{R} : I \mapsto \dim_{L((\mathbb{Z}_2^{\oplus \mathbf{F}_2}/V_I) \rtimes \mathbf{F}_2)} \ker(Q_I - 4)$$

is strictly increasing, where $\mathbf{F}_2 \curvearrowright \mathbb{Z}_2^{\oplus \mathbf{F}_2}/V_I$ by left-coordinate-translation.

Since a strictly increasing map is an injection, the image in $\mathbb{R}$ of $\mathcal{P}(\mathbb{N})$ under this map must be uncountable, and so we immediately obtain the following.

Corollary 1.2 *For some left-translation-invariant subspace $V \leq \mathbb{Z}_2^{\oplus \mathbf{F}_2}$, the finitely-generated group $(\mathbb{Z}_2^{\oplus \mathbf{F}_2}/V) \rtimes \mathbf{F}_2$ admits a group ring element with rational coefficients whose kernel has irrational (and even transcendental) von Neumann dimension.* $\square$

The main innovation of this paper is to exploit the freedom in the choice of the subgroup V above in order to obtain a large family of von Neumann dimensions, some of which must then be irrational, rather than trying to find one single example of a group and group ring element and compute the von Neumann dimension of its kernel explicitly. It is this idea that we will make precise in obtaining the family of examples promised in Theorem 1.1. A similar instance of exploiting this freedom in the choice of V to produce an example of a group with interesting properties appeared recently in [2], and the present paper was indirectly motivated by that one.

Acknowledgements

I am grateful to Dimitri Shlyakhtenko for explaining to me much of the background to Atiyah's question, and to Wolfgang Lück, Peter Linnell and Thomas Schick for insightful suggestions on improving this paper.

2 Some preliminary manipulations

In this section we let Λ be any discrete group and U any discrete Abelian group equipped with a left action $\alpha : \Lambda \curvearrowright U$ by automorphisms (so $\alpha^{gh} = \alpha^g \circ \alpha^h$), and form the semidirect product $U \rtimes_\alpha \Lambda$ as the set-theoretic Cartesian product $U \times \Lambda$ with the multiplication

$$(u, g) \cdot (w, h) := (\alpha^{h^{-1}}(u) + w, gh).$$

We now describe an identification of the left regular action

$$\lambda : \big(\mathbb{C}(U \rtimes_\alpha \Lambda) \subset L(U \rtimes_\alpha \Lambda)\big) \curvearrowright \ell^2(U \rtimes_\alpha \Lambda)$$

that will prove convenient later.

The point is simply that the Fourier transform sets up a unitary isomorphism

$$\mathcal{F} : \ell^2(U) \xrightarrow{\cong} L^2(m_{\widehat{U}})$$

$$\mathcal{F}f(\chi) := \sum_{u\in U} \langle u, \chi\rangle f(u),$$

and now since $U \rtimes_\alpha \Lambda$ is set-theoretically simply equal to $U \times \Lambda$, we have also

$$\mathcal{F}\otimes\mathrm{Id}_{\ell^2(\Lambda)} : \ell^2(U\rtimes_\alpha\Lambda) \cong \ell^2(U)\otimes\ell^2(\Lambda) \xrightarrow{\cong} L^2(m_{\widehat{U}})\otimes\ell^2(\Lambda) \cong L^2(m_{\widehat{U}}\otimes\#_\Lambda),$$

where we write $\#_S$ to denote the counting measure on a set S. Let $\widehat{\alpha} : \Lambda \curvearrowright \widehat{U}$ be the Pontrjagin adjoint action of α defined by the relation

$$\langle u, \widehat{\alpha}^g(\chi)\rangle := \langle \alpha^{g^{-1}}(u), \chi\rangle,$$

and recall that the duality $\langle\cdot,\cdot\rangle : U \times \widehat{U} \to \mathbb{T}$ establishes the Pontrjagin isomorphism $U \cong \widehat{\widehat{U}}$.

As is standard, the isomorphism $\mathcal{F}$ of Hilbert spaces now defines an isomorphism of actions

$$\left(\lambda : \begin{array}{c}\mathbb{C}(U \rtimes_\alpha \Lambda)\\ \cap \\ L(U \rtimes_\alpha \Lambda)\end{array} \curvearrowright \ell^2(U\rtimes_\alpha\Lambda)\right) \xrightarrow{\cong} \left(\pi : \begin{array}{c}\mathbb{C}(U \rtimes_\alpha \Lambda)\\ \cap \\ L^\infty(m_{\widehat{U}}) \rtimes_{\widehat{\alpha}} \Lambda\end{array} \curvearrowright L^2(m_{\widehat{U}}\otimes\#_\Lambda)\right),$$

where π is the left regular action of the group measure space von Neumann algebra $L^\infty(m_{\widehat{U}}) \rtimes_{\widehat{\alpha}} \Lambda$ on $L^2(m_{\widehat{U}} \otimes \#_\Lambda)$, and within $\mathcal{U}\big(L^\infty(m_{\widehat{U}}) \rtimes_{\widehat{\alpha}} \Lambda\big)$ we identify copies of $U \cong \widehat{\widehat{U}}$ and Λ that together generate a copy of $U \rtimes_\alpha \Lambda$ acting by

$$\pi(u,e)f(\chi,g) = \langle\alpha^{g^{-1}}(u),\chi\rangle f(\chi,g) = (M_{\langle u,\cdot\rangle}f)(\chi,g) \qquad \text{for } (\chi,g) \in \widehat{U}\times\Lambda,$$

where M_F denotes twisted pointwise multiplication

$$M_F f(\chi,g) := F(\widehat{\alpha}^g(\chi))f(\chi,g)$$

by a function $F \in L^\infty(m_{\widehat{U}})$, and

$$\pi(0,h)f(\chi,g) = f(\chi,h^{-1}g) =: T^h f(\chi,g) \qquad \text{for } (\chi,g) \in \widehat{U} \times \Lambda,$$

so this is still a translation operator. If $W \subseteq \widehat{U}$ is a Borel subset we will generally write M_W in place of M_{1_W}.

We may double-check that the above specifications do combine to give an action of $U \rtimes_\alpha \Lambda$ through the following commutation relation:

$$\begin{aligned}(T^{h^{-1}} \circ M_F \circ T^h)f(\chi, g) &= ((M_F \circ T^h)f)(\chi, hg) \\ &= F(\widehat{\alpha}^{hg}(\chi)) \cdot (T^h f)(\chi, hg) \\ &= (F \circ \widehat{\alpha}^h)(\widehat{\alpha}^g(\chi)) \cdot f(\chi, g) \\ &= M_{F\circ\widehat{\alpha}^h} f(\chi, g)\end{aligned}$$

for $F \in L^\infty(m_{\widehat{U}})$ and $h \in \Lambda$.

These manipulations lead to a simple identification between group von Neumann algebras $L(U \rtimes_\alpha \Lambda)$ and group measure space algebras $L^\infty(m_{\widehat{U}}) \rtimes_{\widehat{\alpha}} \Lambda$ corresponding to dynamical systems $\widehat{\alpha} : \Lambda \curvearrowright \widehat{U}$ of algebraic origin. In the case $\Lambda = \mathbb{Z}^d$ for $d \geq 2$ such dynamical systems are known to exhibit a wide variety of interesting behaviour (see, in particular, the monograph [13] of Schmidt), and in recent years the analysis of such systems for certain non-Abelian Λ has also begun to make headway (see, for instance, the paper [4] of Deninger and Schmidt and the further references given there). In the present paper we make our own modest appeal to this dynamical picture of semidirect products with Abelian kernel, and it would be interesting to explore whether insights from that field could be used to drive other constructions in geometric group theory in the future.

The above shows that to study the von Neumann algebra properties of $\lambda(\mathbb{Q}(U \rtimes_\alpha \Lambda))$ (turning our attention now to the rational group ring) we may equivalently consider $\pi(\mathbb{Q}(U \rtimes_\alpha \Lambda))$, whose members may all be put into the form

$$\sum_{i=1}^{n} T^{g_i} \circ M_{\phi_i}$$

with $g_i \in \Lambda$ for each $1 \leq i \leq n$ and each $\phi_i \in C(\widehat{U})$ being a trigonometric polynomial with rational coefficients (that is, a finite $\mathbb{Q}$-linear combination of characters) on $\widehat{U}$. It is this form for our operators that will be most convenient for the proof of Theorem 1.1.

We will henceforth apply the above manipulations in case $\Lambda = \mathbf{F}_2$, and will specialize to groups U of the form $\mathbb{Z}_2^{\oplus \mathbf{F}_2}/V$ for some left-translation-invariant subgroup $V \leq \mathbb{Z}_2^{\oplus \mathbf{F}_2}$, equipped with the left translation action

$$\alpha^g((u_h)_{h\in\mathbf{F}_2} + V) = (u_{g^{-1}h})_{h\in\mathbf{F}_2} + V.$$

In this case the Pontrjagin duals obey the relations

$$\widehat{\mathbb{Z}_2^{\oplus \mathbf{F}_2}} \cong \widehat{\mathbb{Z}_2}^{\mathbf{F}_2} \cong \mathbb{Z}_2^{\mathbf{F}_2}$$

and

$$\widehat{\mathbb{Z}_2^{\oplus \mathbf{F}_2}/V} \cong V^\perp := \big\{\boldsymbol{\chi} \in \mathbb{Z}_2^{\mathbf{F}_2} : \langle \mathbf{v}, \boldsymbol{\chi}\rangle = 0\ \forall \mathbf{v} \in V\big\}.$$

We recognize $\widehat{\alpha} : \mathbf{F}_2 \curvearrowright V^\perp$ as the subshift of the left-acting topological Bernoulli shift $\mathbf{F}_2 \curvearrowright \mathbb{Z}_2^{\mathbf{F}_2}$ defined by the relations of annihilating all members of V. Let us also note for future reference that with these conventions, if $\boldsymbol{\chi} = (\chi_h)_{h\in\mathbf{F}_2} \in \mathbb{Z}_2^{\mathbf{F}_2}$ and $g \in \mathbf{F}_2$, then regarding $\boldsymbol{\chi}$ as a colouring of $\mathrm{Cay}(\mathbf{F}_2, S)$ by elements of $\mathbb{Z}_2$, the point $\widehat{\alpha}^{g^{-1}}(\boldsymbol{\chi})$ is obtained by shifting that colouring by the graph automorphism of $\mathrm{Cay}(\mathbf{F}_2, S)$ that moves the point g to the origin and respects the directions of all the edges.

Note also that in this case the rational trigonometric polynomials on $\widehat{U}$ are easily seen to be those functions on $V^\perp$ that are restrictions of functions on $\mathbb{Z}_2^{\mathbf{F}_2}$ that depend on only finitely many coordinates and that take only rational values (using the fact that characters on groups of the form $\mathbb{Z}_2^I$ take only the values ± 1, so in particular are all rational-valued), and so henceforth we will freely work with such functions when specifying members of $\pi(\mathbb{Q}((\mathbb{Z}_2^{\oplus\mathbf{F}_2}/V) \rtimes_\alpha \mathbf{F}_2))$ of interest. We will also now work only with left-translation actions of $\mathbf{F}_2$ such as the above, and so will usually omit their explicit mention from our notation.

3 Introduction of the operators

3.1 Construction

We now specialize to certain particular operators in $\pi(\mathbb{Q}((\mathbb{Z}_2^{\oplus\mathbf{F}_2}/V) \rtimes \mathbf{F}_2))$. These will take the form

$$Q = \sum_{s\in S} T^{s^{-1}} \circ (M_{F_s} + M_{G_s \circ \widehat{\alpha}^{s^{-1}}})$$

where $F_s, G_s : \mathbb{Z}_2^{\mathbf{F}_2} \to \mathbb{Q}$ for $s \in S$ depend only on some finite patch of coordinates around $e \in \mathbf{F}_2$. Note that in considering the above operator as a member of $\pi(\mathbb{Q}((\mathbb{Z}_2^{\oplus\mathbf{F}_2}/V) \rtimes \mathbf{F}_2))$, we are implicitly regarding the above as a shorthand for

$$\sum_{s\in S} T^{s^{-1}} \circ (M_{F_s|_{V^\perp}} + M_{G_s|_{V^\perp} \circ \widehat{\alpha}^{s^{-1}}});$$

we will generally overlook this notational detail in the following.

The rather redundant form in which Q has been written above, with a sum of two terms of the form M_F for each $s \in S$, is convenient in view of the following simple calculation.

Lemma 3.1 *If $F_s = G_{s^{-1}}$ for every $s \in S$ then Q is self-adjoint.*

Proof Since M_F is self-adjoint whenever F takes real values and $(T^s)^* = T^{s^{-1}}$, we deduce from the commutator relation for these operators, the symmetry of S and our assumption that

$$\begin{aligned} Q^* &= \sum_{s\in S}(M_{F_s} + M_{G_s \circ \widehat{\alpha}^{s^{-1}}}) \circ T^s \\ &= \sum_{s\in S} T^s \circ (M_{F_s \circ \widehat{\alpha}^s} + M_{G_s \circ \widehat{\alpha}^{s^{-1}} \circ \widehat{\alpha}^s}) \\ &= \sum_{s\in S} T^s \circ (M_{G_{s^{-1}} \circ \widehat{\alpha}^s} + M_{F_{s^{-1}}}) = Q. \end{aligned}$$

□

Most of this section will be concerned with the choice of F_s and G_s, which will be pivotal for what follows. We will choose functions that depend only on coordinates in the ball $B(e, 100)$. Heuristically, the values of $F_s(\boldsymbol{\chi})$ will depend on different features of the level-set $\boldsymbol{\chi}^{-1}\{0\}$ describing in what ways it locally resembles a path in $\mathrm{Cay}(\mathbf{F}_2, S)$, what that path looks like, and whether it contains e. To explain this we first make the following useful definitions.

Definition 3.2 (Small horizontal doglegs) *A (finite or infinite) path $P \subset \mathrm{Cay}(\mathbf{F}_2, S)$ contains a **small horizontal dogleg** if it contains a subset of the form*

$$\{gs_2^{\eta'}, g, gs_1^{\eta}, gs_1^{2\eta}, \dots, gs_1^{\ell\eta}, gs_1^{\ell\eta}s_2^{\eta''}\} \qquad \text{for some } g \in \mathbf{F}_2,\ \ell \in \{1, 2, \dots, 9\},\ \eta', \eta, \eta'' \in \{-1, 1\}$$

or of the form

$$\{g, gs_1^{\eta}, gs_1^{2\eta}, \dots, gs_1^{\ell\eta}, gs_1^{\ell\eta}s_2^{\eta''}\} \qquad \text{for some } g \in \mathbf{F}_2,\ \ell \in \{1, 2, \dots, 9\},\ \eta, \eta'' \in \{-1, 1\}, \text{ with } g \text{ an end-point of } P.$$

*(note that only the first of these cases really fits the term 'dogleg'). Otherwise P contains **no small horizontal doglegs**. In either case we refer to the further subset $\{g, gs_1^{\eta}, gs_2^{2\eta}, \dots, gs_1^{\ell\eta}\}$ as the **main segment** of the dogleg.*

Definition 3.3 (Locally good points) *A point $\boldsymbol{\chi} \in \mathbb{Z}_2^{\mathbf{F}_2}$ is **locally good** if*

1. *$\boldsymbol{\chi}^{-1}\{0\} \cap B(e, 10)$ is a path in $\mathrm{Cay}(\mathbf{F}_2, S)|_{B(e,10)}$ that contains e and has length at least 10 (that is, it connects e with some point of $\partial B(e, 9) \subset B(e, 10)$),*

2. *there is no small horizontal dogleg in the path* $\boldsymbol{\chi}^{-1}\{0\} \cap B(e, 10)$ *whose main segment lies within* $B(e, 9)$*, and*

3. *for every* $g \in \boldsymbol{\chi}^{-1}\{0\} \cap B(e, 10)$ *we also have that* $\boldsymbol{\chi}^{-1}\{0\} \cap B(g, 10)$ *is a path in* $\mathrm{Cay}(\mathbf{F}_2, S)|_{B(g,10)}$ *containing no small horizontal doglegs with main segment contained in* $B(g, 9)$*.*

The second part of the above definition is very important. It places rather severe restrictions on which paths can appear as $\boldsymbol{\chi}^{-1}\{0\} \cap B(e, 10)$ if $\boldsymbol{\chi}$ is locally good: insofar as a path in $\mathrm{Cay}(\mathbf{F}_2, S)$ is made up of a concatenation of 'horizontal' segments (with steps given by $s_1^{\pm 1}$) and 'vertical' segments (with steps given by $s_2^{\pm 1}$), this condition tells us that while the maximal vertical segments that appear in $\boldsymbol{\chi}^{-1}\{0\} \cap B(e, 10)$ may be of any length, this path may not contain any maximal horizontal segments that lie properly inside $B(e, 10)$ and have length less than 10. It follows that if a maximal horizontal segment lies properly inside $B(e, 10)$ (that is, the end-points of that segment also visibly lie inside $B(e, 10)$), then it must contain e as an interior point and extend to points s_1^a and s_1^{-b} with $a, b \geq 1$ and $a + b \geq 10$, before being permitted to make at most one more horizontal-vertical-horizontal 'dogleg' before leaving $B(e, 10)$ on either side of e. Moreover, the last condition of the above definition ensures that not only does the vertex e see this highly constrained behaviour in its radius-10 neighbourhood, but also all of its neighbours inside this path and at distance at most 10 see this behaviour in their radius-10 neighbourhoods. This rather peculiar restriction on the kinds of path we allow will be pivotal at exactly one point below (Corollary 5.7), where it will restrict a certain sum over paths to terms that possess some additional helpful properties.

We will give a definition of F_s (and then set $G_s = F_{s^{-1}}$) that uses the above notion, but we first define another auxiliary function F_s°.

Definition 3.4 *The function* $F_s^\circ : \mathbb{Z}_2^{\mathbf{F}_2} \to \mathbb{Q}$ *is defined according to the following four cases:*

- $F_s^\circ(\boldsymbol{\chi}) := 1$ *if* $\boldsymbol{\chi}$ *is locally good and* e *and* s *are both interior points of the path* $\boldsymbol{\chi}^{-1}\{0\} \cap B(e, 10)$*;*

- $F_s^\circ(\boldsymbol{\chi}) := 2$ *if* $\boldsymbol{\chi}$ *is locally good,* e *is an interior point of the path* $\boldsymbol{\chi}^{-1}\{0\} \cap B(e, 10)$ *and* s *is its end-point; or if* $\boldsymbol{\chi}$ *is locally good and the path* $\boldsymbol{\chi}^{-1}\{0\} \cap B(e, 10)$ *contains both* e *and also some* $t \in S \setminus \{s, s^{-1}\}$*, but does not contain* s*;*

- $F_s^\circ(\boldsymbol{\chi}) := \frac{1}{100}$ *if* $\boldsymbol{\chi}$ *is* not *locally good, but we do have that* $e \in \boldsymbol{\chi}^{-1}\{0\}$ *and that the translate* $\widehat{\alpha}^{s^{-1}}(\boldsymbol{\chi})$ *is locally good.*
- $F_s^\circ(\boldsymbol{\chi}) := 0$ *otherwise.*

Remarks 1. In particular, $F_s^\circ(\boldsymbol{\chi}) = 0$ unless $e \in \boldsymbol{\chi}^{-1}\{0\}$ and $\boldsymbol{\chi}^{-1}\{0\} \cap B(e,10)$ is a path in $\mathrm{Cay}(\mathbf{F}_2, S)|_{B(e,10)}$, and given these conditions the exact value of $F_s^\circ(\boldsymbol{\chi})$ is determined by a further sub-classification.

2. Let us draw attention to the quirk that if $\boldsymbol{\chi}$ is locally good but e is an *end-point* of the path $\boldsymbol{\chi}^{-1}\{0\} \cap B(e,10)$ with neighbour s also lying in this path, then

$$\begin{array}{rcl} F_s(\boldsymbol{\chi}) & = & F_{s^{-1}}(\boldsymbol{\chi}) = 0, \\ G_s(\widehat{\alpha}^{s^{-1}}(\boldsymbol{\chi})) & = & F_{s^{-1}}(\widehat{\alpha}^{s^{-1}}(\boldsymbol{\chi})) = 2 \\ \text{and} \quad F_t(\boldsymbol{\chi}) & = & 2 \quad \text{for } t \in S \setminus \{s, s^{-1}\}. \end{array}$$

This slightly tricky case will give rise to a useful simplification later.

3. In the third case above we must have that $\boldsymbol{\chi}^{-1}\{0\} \cap B(e,10)$ is a path containing e, so this case can arise only because there is some point $g \in \boldsymbol{\chi}^{-1}\{0\}$ that lies at distance 10 from e and 11 from s, such that g also lies at distance 10 from some 'bad' feature of $\boldsymbol{\chi}^{-1}\{0\}$ — a fork, a distinct connected component, or a small horizontal dogleg visible in its entirety — so that some other condition in the definition of 'locally good' is violated. It is easy to see that in this scenario there can be only one such s, since if $s' \in S$ were another then the path $\boldsymbol{\chi}^{-1}\{0\} \cap B(e,10)$ would have to contain both s and s', and so must connect them via e, but in this case we see easily from the definition that if both $\widehat{\alpha}^{s^{-1}}(\boldsymbol{\chi})$ and $\widehat{\alpha}^{(s')^{-1}}(\boldsymbol{\chi})$ are locally good then so is $\boldsymbol{\chi}$.

4. Of course, the particular value $\frac{1}{100}$ employed in the third case above is not very important; it has been chosen simply as a rational number that will easily be shown to satisfy a certain modest algebraic condition that we need later. $\lhd$

We also let $G_{s^{-1}}^\circ := F_s^\circ$, and now note the following consequence of this definition.

Lemma 3.5 *For any* $\boldsymbol{\chi} \in \mathbb{Z}_2^{\mathbf{F}_2}$ *the set*

$$E(\boldsymbol{\chi}) := \bigcup_{s \in S} \big\{ g \in \mathbf{F}_2 : \ F_s^\circ(\widehat{\alpha}^{g^{-1}}(\boldsymbol{\chi})) \text{ and } G_s^\circ(\widehat{\alpha}^{s^{-1}g^{-1}}(\boldsymbol{\chi})) \text{ not both } 0 \big\}$$

is a union of connected components in $\mathrm{Cay}(\mathbf{F}_2, S)$ *each of which takes the form* $B(P,1) \setminus T$ *for some path* P *with no small horizontal doglegs and some set* T *of*

at most two boundary points of end-points of P, and any two of these connected components are separated by a distance of at least 9.

Proof If $F_s^\circ(\widehat{\alpha}^{g^{-1}}(\boldsymbol{\chi}))$ and $G_s^\circ(\widehat{\alpha}^{s^{-1}g^{-1}}(\boldsymbol{\chi}))$ are not both zero, then from the first remark above it follows that either g is itself a member of $\boldsymbol{\chi}^{-1}\{0\}\cap B(g,10)$ and $\widehat{\alpha}^{g^{-1}}(\boldsymbol{\chi})$ is locally good, so this set takes the form of a path with no small horizontal doglegs in $\mathrm{Cay}(\mathbf{F}_2,S)|_{B(g,10)}$, or g is adjacent to such a point. Clearly these paths in balls of radius 10 patch together to form, together with their immediate neighbourhoods, the connected components of the given set, so each of these must be a path in $\mathrm{Cay}(\mathbf{F}_2,S)$ together with all but possibly two members of its neighbourhood (these being precisely the points such as s^{-1} in the situation described in Remark 2 above). From the definition of F_s it follows that any two distinct such paths must lie at a distance of at least 11 from each other (and so their radius-1 neighbourhoods must lie at distance at least 9), in order that points internal to these paths should not see foreign connected components within their radius-10 neighbourhoods. $\square$

Corollary 3.6 *The set*

$$W := \big\{\boldsymbol{\chi}\in\mathbb{Z}_2^{\mathbf{F}_2}:\ E(\boldsymbol{\chi})\ni e \text{ but the central path of the component that contains } e \text{ has length } \leq 4\big\}$$

depends only on coordinates in $B(e,100)$. $\square$

Finally, we set

$$F_s := F_s^\circ\cdot 1_W \qquad G_s := F_{s^{-1}} = G_s^\circ\cdot 1_W$$

and consider the resulting operator Q, which by Lemma 3.1 is self-adjoint.

3.2 Decomposition into invariant subspaces

In describing the further consequences of our choice of F_s the following terminology will prove convenient.

Definition 3.7 (Good and bad neighbourhoods) *For a given point $\boldsymbol{\chi}\in\mathbb{Z}_2^{\mathbf{F}_2}$, a ball $B(g,10)\subset\mathbf{F}_2$ is a* ***good*** *(respectively,* ***bad****)* ***neighbourhood for $\boldsymbol{\chi}$*** *if $\widehat{\alpha}^{g^{-1}}(\boldsymbol{\chi})$ is locally good and g is an end-point of the path $\boldsymbol{\chi}^{-1}\{0\}\cap B(g,10)$ (respectively, if $\widehat{\alpha}^{g^{-1}}(\boldsymbol{\chi})$ is not locally good, but $g\in\boldsymbol{\chi}^{-1}\{0\}$ and for some $s\in S$ the translate $\widehat{\alpha}^{s^{-1}g^{-1}}(\boldsymbol{\chi})$ is locally good).*

Now consider a point $\boldsymbol{\chi} \in \mathbb{Z}_2^{\mathbf{F}_2}$: either there is some $s \in S$ such that

$$F_s(\boldsymbol{\chi}) \text{ and } G_s(\widehat{\alpha}^{s^{-1}}(\boldsymbol{\chi})) \text{ are not both } 0,$$

or there is not. Let C_0 be the set of those $\boldsymbol{\chi}$ for which there is not; this is clearly a clopen subset of $\boldsymbol{\chi}$. Our next step will be to obtain a rather more detailed partition of the remainder $\mathbb{Z}_2^{\mathbf{F}_2} \setminus C_0$.

Thus, suppose now that $\boldsymbol{\chi} \in \mathbb{Z}_2^{\mathbf{F}_2} \setminus C_0$, and that $s \in S$ is such that $F_s(\boldsymbol{\chi}) \neq 0$. It follows that either $\boldsymbol{\chi}$ is locally good, or (if $F_s(\boldsymbol{\chi}) = \frac{1}{100}$) that $e \in \boldsymbol{\chi}^{-1}\{0\}$ and $\widehat{\alpha}^{s^{-1}}(\boldsymbol{\chi})$ is locally good. In either case this requires that $\boldsymbol{\chi}^{-1}\{0\} \cap B(e, 10)$ be a path with no small horizontal doglegs that passes through e.

Similarly, if $G_s(\widehat{\alpha}^{s^{-1}}(\boldsymbol{\chi})) \neq 0$, then either $\boldsymbol{\chi}$ is locally good and so $\boldsymbol{\chi}^{-1}\{0\} \cap B(e, 10)$ is a path that passes through e, or $\widehat{\alpha}^{s^{-1}}(\boldsymbol{\chi})$ is locally good and $\boldsymbol{\chi}^{-1}\{0\} \cap B(e, 9)$ is a path containing s but no other member of $S \cup \{e\}$.

In either of the above cases we may pick a unique $g_0 \in S \cup \{e\}$ that is closest to e and such that $\widehat{\alpha}^{g_0^{-1}}(\boldsymbol{\chi})$ is locally good.

Now imagine dispatching two walkers from g_0 towards the two different end-points of the path $\boldsymbol{\chi}^{-1}\{0\} \cap B(g_0, 10)$ with instructions to walk in their given directions along edges that remain in the level set $\boldsymbol{\chi}^{-1}\{0\}$ and through vertices g such that $\widehat{\alpha}^{g^{-1}}(\boldsymbol{\chi})$ is still locally good, until they reach either a good neighbourhood or a bad neighbourhood for $\boldsymbol{\chi}$, where they should stop and report back to us. It may happen that one or both of them leave $B(g_0, 10)$, or that they do not move at all.

If a walker never reaches a good or bad neighbourhood, then it follows that the level set $\boldsymbol{\chi}^{-1}\{0\}$ that she followed in her direction must continue to look like a path, with no end-points, forks, small horizontal doglegs or distinct components lying within distance 10 of it: otherwise the walker would at some point have stopped walking in a bad neighbourhood. Let us call this walking-forever scenario (∞).

If the walker reaches a good neighbourhood, then she has followed a path-like branch of $\boldsymbol{\chi}^{-1}\{0\}$ with no small horizontal doglegs until reaching an end-point of that path, and again this finite-length path-like branch has no other points of $\boldsymbol{\chi}^{-1}\{0\}$ lying within distance 10 of it. Note that this includes the possibility that g_0 is an end-point of the path, and so this walker is already in a good neighbourhood initially. We call this ending scenario (1).

The final scenario, that the walker's journey terminates in a bad neighbourhood, may result from three different features of $\boldsymbol{\chi}^{-1}\{0\}$: a point of this level set not connected to the walker's path, but lying within distance 10 of it; a fork in the

path; or a small horizontal dogleg. In any case the walker stop walking as soon as he reaches within distance 10 of some further point of his path which, in turn, can see this feature within its radius-10 neighbourhood (effectively he has had a premonition of this bad feature within distance 10 of his own radius-10 horizon). This rather convoluted description is important, because it causes this walker to stop far short of actually reaching, or even being himself able to see, this non-path-like feature (rather than, for example, continuing until he actually reaches a fork), and we will find that this greatly simplifies certain enumerations later. Note that in case $g_0 \neq e$, this includes the possibility that this walker is dispatched from g_0 back towards e, then reaches e where this happens and stops. We call this ending scenario (2).

Finally, note also that from the definition of F_s as $F_s^\circ \cdot 1_W$, the combined distances walked by the two walkers must be at least 5; this rules out some annoying degenerate scenarios, and was why we introduced the set W.

Now, every point $\boldsymbol{\chi} \in \mathbb{Z}_2^{\mathbf{F}_2} \setminus C_0$ results in a pair of ending scenarios, each from the set $\{(1),(2),(\infty)\}$, according to the fates of the two walkers. Together their route specifies some (finite or infinite) path $P \subseteq \boldsymbol{\chi}^{-1}\{0\}$. Regarding the two walkers as indistinguishable except by their ending scenarios, we can now partition $\mathbb{Z}_2^{\mathbf{F}_2} \setminus C_0$ into the six (manifestly Borel) sets $C_{a,b}$ for $a, b \in \{1, 2, \infty\}$ and $a \leq b$, where $\boldsymbol{\chi} \in C_{a,b}$ if one walker ends in scenario (a) and the other in scenario (b). Also, if either walker ends in a bad neighbourhood, then we know that the path they were following extends another 10 steps beyond their ending position to a point that can see bad behaviour within distance 10 of itself, and so including these last few steps if available defines a larger path $R \subseteq \boldsymbol{\chi}^{-1}\{0\}$, $R \supseteq P$ (which respectively equals P or extends it at one or both of its end-points according as $\boldsymbol{\chi} \in C_{1,1} \cup C_{1,\infty}$, $C_{1,2} \cup C_{2,\infty}$ or $C_{2,2}$).

Thus we have obtained the Borel partition

$$\mathbb{Z}_2^{\mathbf{F}_2} = C_0 \cup C_{1,1} \cup C_{1,2} \cup C_{2,2} \cup C_{1,\infty} \cup C_{2,\infty} \cup C_{\infty,\infty}.$$

In fact, it is easy to refine this partition even further. If $\boldsymbol{\chi} \in C_{a,b}$ with $a, b < \infty$, then P and R are finite subsets of $\mathbf{F}_2$. Moreover, the fact that $\boldsymbol{\chi} \in C_{a,b}$ now depends only on the restriction $\boldsymbol{\chi}|_{B(R,10)}$ (in the sense that any other $\boldsymbol{\chi}'$ agreeing with $\boldsymbol{\chi}$ on this restriction also lies in $C_{a,b}$, with walkers seeing just the same configurations). We may therefore partition $C_{a,b}$ according to the triples (P, R, ψ), where $\psi := \boldsymbol{\chi}|_{B(R,10)\setminus R}$, that can arise in this way.

Let $\Omega_{a,b}$ be the collection of triples (P, R, ψ) such that any point $\boldsymbol{\chi}$ giving rise to them as above must lie in $C_{a,b}$, and let

$$C_{P,R,\psi} := \{\boldsymbol{\chi} \in \mathbb{Z}_2^{\mathbf{F}_2} :\ \boldsymbol{\chi}^{-1}\{0\} \supseteq R \text{ and } \boldsymbol{\chi}|_{B(R,10)\setminus R} = \psi\}$$

be the cylinder set associated to this triple. In this situation we will refer to P as the **inner path** and R as the **outer path** of (P, R, ψ). Clearly $R = P$ if and only if $a = b = 1$, and sometimes we will abusively write members of $\Omega_{1,1}$ simply as pairs (P, ψ).

We have now obtained the following finer partition.

Lemma 3.8 *The equality*

$$\mathbb{Z}_2^{\mathbf{F}_2} = C_0 \cup \Big(\bigcup_{\substack{a, b \in \{1,2\}, \\ a \leq b}} \bigcup_{(P,R,\psi) \in \Omega_{a,b}} C_{P,R,\psi} \Big) \cup C_{1,\infty} \cup C_{2,\infty} \cup C_{\infty,\infty}$$

holds, and is a Borel partition of $\mathbb{Z}_2^{\mathbf{F}_2}$. □

From this partition we can obtain a related orthogonal decomposition of the Hilbert space $L^2(m_{V^\perp} \otimes \#_{\mathbf{F}_2})$, and it in this form that its importance will become clear: we will later obtain a simple description of Q in terms of its behaviour on each of these subspaces that will then enable us to identify certain of its eigenspaces exactly. For each $(P, R, \psi) \in \Omega_{a,b}$ we define

$$\mathfrak{H}_{P,R,\psi} := \mathrm{img}(M_{C_{P,R,\psi}})$$

and also

$$\mathfrak{H}_0 := \mathrm{img}(M_{C_0}) \qquad \text{and} \qquad \mathfrak{H}_{a,\infty} := \mathrm{img}(M_{C_{a,\infty}}) \qquad \text{for } a \in \{1, 2, \infty\},$$

and so now we can write

$$\begin{aligned} &L^2(m_{V^\perp} \otimes \#_{\mathbf{F}_2}) \\ &\quad = \mathfrak{H}_0 \oplus \Big(\bigoplus_{\substack{a, b \in \{1,2\}, \\ a \leq b}} \bigoplus_{(P,R,\psi) \in \Omega_{a,b}} \mathfrak{H}_{P,R,\psi} \Big) \oplus \mathfrak{H}_{1,\infty} \oplus \mathfrak{H}_{2,\infty} \oplus \mathfrak{H}_{\infty,\infty}. \end{aligned}$$

Note that since each component of this decomposition is defined by an orthogonal projection lying in the von Neumann algebra $L^\infty(m_{V^\perp}) \rtimes \mathbf{F}_2 \cong L\Gamma$, each defines a submodule for the right action of Γ on $L^2(m_{V^\perp} \otimes \#_{\mathbf{F}_2})$ (arising by applying the Fourier transform to the right von Neumann algebra $R\Gamma$ acting on $\ell^2(\Gamma)$, as in Section 2), and has a well-defined von Neumann dimension given by the standard trace on $L^\infty(m_{V^\perp}) \rtimes \mathbf{F}_2$.

It will turn out that for a suitable choice of $V^\perp$ we have

$$m_{V^\perp}(C_{a,\infty}) = 0 \qquad \forall a \in \{1, 2, \infty\},$$

so that the spaces $\mathfrak{H}_{a,\infty}$ for $a \in \{1, 2, \infty\}$ contribute trivially to the above decomposition. This will be proved in Proposition 5.8 once we have specified our method for choosing V. In the remainder of this section we make a closer examination of the behaviour of Q on the subspaces $\mathfrak{H}_0$ and $\mathfrak{H}_{P,R,\psi}$.

We first organize the above orthogonal decomposition by 'clustering' the subspaces involved into certain equivalence classes, in such a way that the subspaces of the coarser decomposition that results from this clustering are individually Q-invariant and each admits a relatively simple description of the action of Q. The equivalence relation we need is the following.

Definition 3.9 (Translation equivalence) *Two triples (P_1, R_1, ψ_1) and (P_2, R_2, ψ_2), with P_1, P_2 (finite or infinite) paths in $\mathbf{F}_2$ that pass within distance 1 of e and $\psi_i : B(R_i, 10) \setminus R_i \to \mathbb{Z}_2$, are **translation equivalent** (denoted by $(P_1, R_1, \psi_1) \sim (P_2, R_2, \psi_2)$) if there is some $g \in \mathbf{F}_2$ such that $P_2 = gP_1$, $R_2 = gR_1$ and $\psi_2(gh) = \psi_1(h)$ for all $h \in B(R_1, 10) \setminus R_1$. In this case we will also write that (P_1, R_1, ψ_1) is a **translate** of (P_2, R_2, ψ_2). Since P_1 and P_2 are both required to pass within distance 1 of e, if P_1 is finite then clearly the equivalence class of (P_1, R_1, ψ_1) is a finite set of size $|B(P_1, 1)|$.*

We use this to re-organize the above orthogonal decomposition as

$$L^2(m_{V^\perp} \otimes \#_{\mathbf{F}_2}) = \mathfrak{H}_0 \oplus \Big(\bigoplus_{\substack{a,b \in \{1,2\} \\ a \le b}} \bigoplus_{\mathcal{C} \in \Omega_{a,b}/\sim} \mathfrak{H}_{\mathcal{C}} \Big) \oplus \mathfrak{H}_{1,\infty} \oplus \mathfrak{H}_{2,\infty} \oplus \mathfrak{H}_{\infty,\infty},$$

where

$$\mathfrak{H}_{\mathcal{C}} := \bigoplus_{(P,R,\psi) \in \mathcal{C}} \mathfrak{H}_{P,R,\psi}.$$

The following is a straightforward extension of Equation (3.5) in Dicks and Schick [5].

Lemma 3.10 *For any $g \in \mathbf{F}_2$ and any Borel subset $Y \subseteq \mathbb{Z}_2^{\mathbf{F}_2}$ we have*

$$Q \circ T^{g^{-1}} \circ M_Y = \sum_{s \in S} T^{s^{-1}g^{-1}} \circ (M_{(F_s \circ \widehat{\alpha}^{g^{-1}}) \cdot 1_Y} + M_{(G_s \circ \widehat{\alpha}^{s^{-1}g^{-1}}) \cdot 1_Y}).$$

$\square$

Lemma 3.11 *We have $Q|_{\mathfrak{H}_0} = 0$.*

Proof By the definition of C_0 and Lemma 3.10 we have

$$F_s \cdot 1_{C_0} = (G_s \circ \widehat{\alpha}^{s^{-1}}) \cdot 1_{C_0} = 0 \qquad \forall s \in S$$

and so

$$\begin{aligned} & M_{C_0} f = f \\ \Rightarrow \qquad & Qf = (Q \circ M_{C_0}) f = \sum_{s \in S} \big(T^{s^{-1}} \circ (M_{F_s \cdot 1_{C_0}} + M_{(G_s \circ \widehat{\alpha}^{s^{-1}}) \cdot 1_{C_0}}) \big) f = 0. \end{aligned}$$

$\square$

Proposition 3.12 *Let $(V^{\ell,a,b}, E^{\ell,a,b})$ for $a, b \in \{1, 2\}$ be the weighted graphs shown in Figure 1 and*

$$Q^{\ell,a,b} = (q^{\ell,a,b}_{u,v})_{u,v \in V^{\ell,a,b}}$$

their weighted adjacency matrices, regarded as operators on $\ell^2(V^{\ell,a,b})$, which we interpret as a trivial von Neumann right-module for $L((\mathbb{Z}_2^{\oplus \mathbf{F}_2}/V) \rtimes \mathbf{F}_2)$. Then for each $\mathcal{C} \in \Omega_{a,b}/\sim$ such that $(P, R, \psi) \in \mathcal{C}$ has $|P| = \ell$, the subspace $\mathfrak{H}_{\mathcal{C}}$ is Q-invariant, and there is some von Neumann right-module $\mathfrak{h}_{\mathcal{C}}$ (which will in fact depend on the measure $m_{V^\perp}$) such that we have

$$Q|_{\mathfrak{H}_{\mathcal{C}}} \cong \mathrm{id}_{\mathfrak{h}_{\mathcal{C}}} \otimes Q^{\ell,a,b}.$$

Proof We treat the case of $Q|_{\mathfrak{H}_{\mathcal{C}}}$ for some $\mathcal{C} \in \Omega_{1,2}/\sim$, the others being similar.

Pick a representative $(P, R, \psi) \in \mathcal{C}$, say with $|P| = \ell$, such that e is the 'good' end-point of P: that is, such that $\boldsymbol{\chi}$ itself is locally good. There is exactly one such end-point if $(P, R, \psi) \in \Omega_{1,2}$. Let $\mathfrak{h}_{\mathcal{C}}$ be the von Neumann right-module $\mathfrak{H}_{P,R,\psi}$ (of course, the dimension of this depends on $m_{V^\perp}$). Owing to the involvement of W in the definition of F_s and hence of $\Omega_{1,2}$, we know that $\ell \geq 5$.

Next observe that if $g \in B(P, 1)$, then the values

$$F_s(\widehat{\alpha}^{g^{-1}}(\boldsymbol{\chi})) \quad \text{and} \quad G_s(\widehat{\alpha}^{s^{-1}} \widehat{\alpha}^{g^{-1}}(\boldsymbol{\chi})) = F_{s^{-1}}(\widehat{\alpha}^{(gs)^{-1}}(\boldsymbol{\chi}))$$

are the same for all $\boldsymbol{\chi} \in C_{P,R,\psi}$. In view of this we can define

$$\phi(g, gs) := F_s(\widehat{\alpha}^{g^{-1}}(\boldsymbol{\chi}))$$

for $g \in B(P, 1)$ using any representative $\boldsymbol{\chi} \in C_{P,R,\psi}$, and obtain

$$\begin{aligned} M_{(F_s \circ \widehat{\alpha}^{g^{-1}}) \cdot 1_{C_{P,R,\psi}}} + M_{(G_s \circ \widehat{\alpha}^{s^{-1} g^{-1}}) \cdot 1_{C_{P,R,\psi}}} \\ = (\phi(g, gs) + \phi(gs, g)) \cdot M_{C_{P,R,\psi}}. \end{aligned} \quad (1)$$

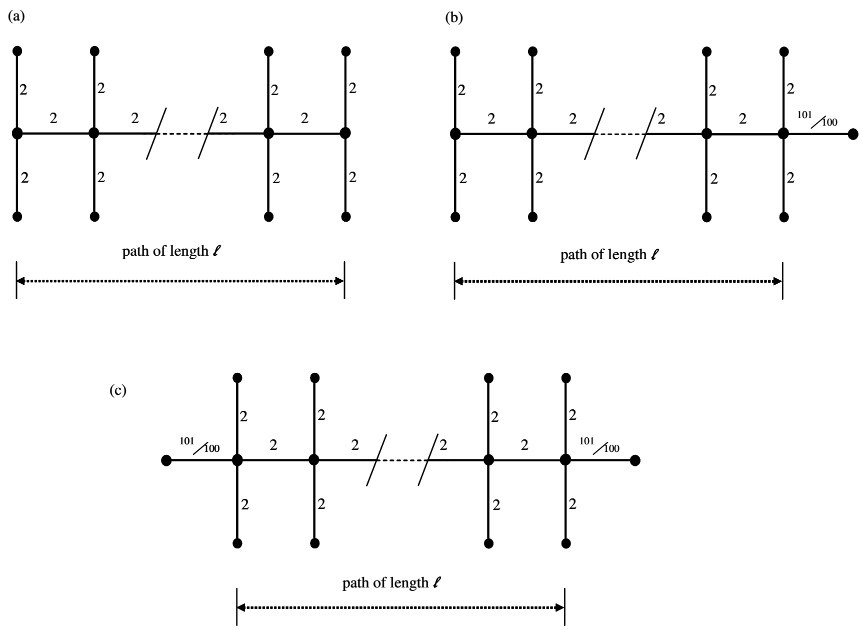

Figure 1: The weighted graph $(V^{\ell,a,b}, E^{\ell,a,b,})$ corresponding to $Q|_{\mathfrak{H}_{\mathcal{C}}}$ for (a) $\mathcal{C} \in \Omega_{1,1}/\sim$, (b) $\mathcal{C} \in \Omega_{1,2}/\sim$ and (c) $\mathcal{C} \in \Omega_{2,2}/\sim$

We can now simply read off from the definition of F_s a very explicit description of this function ϕ on the set of pairs

$$\{(g,h):\ g,h\in B(P,1),\ \rho(g,h)=1\}:$$

- If g is an interior point of P, then it has
  - two neighbours h that are not in P, and for each of these we have $\phi(g,h)=2$ and $\phi(h,g)=0$, so $\phi(g,h)+\phi(h,g)=2$, and
  - two neighbours h that are also in P, so if such an h is also an interior point then $\phi(g,h)=\phi(h,g)=1$ and if it is an end-point of P then $\phi(g,h)=2$ and $\phi(h,g)=0$, and in either case overall $\phi(g,h)+\phi(h,g)=2$;
- If $g=e$ is the good end-point, then it has
  - one neighbour s that must lie in the interior of P, for which $\phi(e,s)=0$ and $\phi(s,e)=2$ and so $\phi(e,s)+\phi(s,e)=2$,
  - an opposite neighbour s^{-1}, for which $\phi(e,s^{-1})=\phi(s^{-1},e)=0$, so $\phi(e,s^{-1})+\phi(s^{-1},e)=0$, and
  - two neighbours t neither of which lie in P and such that e is their mid-point, for each of which $\phi(e,t)=2$ and $\phi(t,e)=0$ so that $\phi(e,t)+\phi(t,e)=2$;
- If g is the other ('bad') end-point of P, so that it still lies in the interior of R, then it has
  - one neighbour h that lies in the interior of P, for which $\phi(g,h)=1$ and $\phi(h,g)=1$ and so $\phi(g,h)+\phi(h,g)=2$,
  - one neighbour h that lies in $R\backslash P$, for which $\phi(g,h)=1$ and $\phi(h,g)=\frac{1}{100}$, so $\phi(g,h)+\phi(h,g)=\frac{101}{100}$, and
  - two neighbours h which do not lie in R, for each of which $\phi(g,h)=2$ and $\phi(h,g)=0$ so that $\phi(g,h)+\phi(h,g)=2$.

Note that the cases above involving the 'good' end-point are where we have used the quirk in the definition of F_s discussed in Remark 2 after Definition 3.4.

Putting these possibilities together, and comparing them with Figure 1(b), we see that if we let $V_0\subseteq V^{\ell,1,2}$ be the subset of ℓ vertices on the central path of that graph then we may choose a bijection $\xi_0:V_0\to P$ such that the left (respectively

right) end-point of V_0 is sent to e (respectively, to the 'bad' end-point of P), and now extend this to an isomorphism of weighted graphs

$$\begin{aligned}\xi : (&V^{\ell,1,2}, E^{\ell,1,2}, Q^{\ell,1,2}) \\ &\to \big(B(P,1), \mathrm{Cay}(\mathbf{F}_2,S)|_{B(P,1)}, (\phi(g,h)+\phi(h,g))_{g,h\in B(P,1),\rho(g,h)=1}\big),\end{aligned}$$

(where we have been just a little sloppy, in that we allow our 'isomorphism of weighted graphs' to miss the isolated neighbour of e with no positive-weight connections). That this is possible follows by inspection of Figure 1(b) and the list of possibilities above, which shows that for each $v \in V_0$ we may pair up its neighbours with those of $\xi_0(v) \in P$ so as to respect the edge-weights:

$$\{u,v\} \in E^{\ell,1,2} \qquad \Rightarrow \qquad \phi(\xi(u),\xi(v)) + \phi(\xi(v),\xi(u)) = q^{\ell,1,2}_{u,v}.$$

We can now simply turn this isomorphism of weighted graphs into an isomorphism of Hilbert space operators as follows. Let $(\delta_v)_{v\in V^{\ell,1,2}}$ be the standard basis of $\ell^2(V^{\ell,1,2})$. Observe from the definition of translation equivalence that

$$\mathcal{C} = \{(gP, gR, \psi(g\,\cdot\,)) :\ g \in B(P,1)\}$$

and that $C_{gP,gR,\psi(g\,\cdot\,)} = \widehat{\alpha}^{g^{-1}}(C_{P,R,\psi})$, and hence that

$$\begin{aligned}\mathfrak{H}_{\mathcal{C}} = \bigoplus_{(P',R',\psi')\sim(P,R,\psi)} \mathfrak{H}_{P',R',\psi'} &= \bigoplus_{g\in B(P,1)} \mathrm{img}\big(M_{\widehat{\alpha}^{g^{-1}}(C_{P,R,\psi})}\big) \\ &= \bigoplus_{g\in B(P,1)} \mathrm{img}\big(T^{g^{-1}} \circ M_{C_{P,R,\psi}} \circ T^g\big) = \bigoplus_{g\in B(P,1)} T^{g^{-1}}(\mathfrak{h}_{\mathcal{C}})\end{aligned}$$

Now define

$$\Phi : \mathfrak{H}_{\mathcal{C}} \to \mathfrak{h}_{\mathcal{C}} \otimes \ell^2(V^{\ell,1,2})$$

by setting

$$\Phi(f) = T^g(f) \otimes \delta_{\xi^{-1}(g)} \qquad \text{for } g \in B(P,1),\ f \in T^{g^{-1}}(\mathfrak{h}_{\mathcal{C}}).$$

This is clearly an isomorphism of von Neumann right-modules, and it is now simple to check that $Q|_{\mathfrak{H}_{\mathcal{C}}} = \Phi^{-1} \circ (\mathrm{id}_{\mathfrak{h}_{\mathcal{C}}} \otimes Q^{\ell,1,2}) \circ \Phi$: indeed, if $f \in T^{g^{-1}}(\mathfrak{h}_{\mathcal{C}})$,

so $M_{C_{P,R,\psi}}(T^g f) = T^g f$, then using Lemma 3.10 and equation (1) we have

$$\begin{aligned}
Qf &= (Q \circ T^{g^{-1}})(T^g f) = (Q \circ T^{g^{-1}})(M_{C_{P,R,\psi}}(T^g f)) \\
&= (Q \circ T^{g^{-1}} \circ M_{C_{P,R,\psi}})(T^g f) \\
&= \sum_{s \in S} (\phi(g, gs) + \phi(gs, g)) \cdot T^{s^{-1}g^{-1}} \circ M_{C_{P,R,\psi}}(T^g f) \\
&= \sum_{s \in S} (\phi(g, gs) + \phi(gs, g)) \cdot T^{s^{-1}g^{-1}}(T^g f) \\
&= \sum_{s \in S} T^{s^{-1}g^{-1}}(q^{\ell,1,2}_{\xi^{-1}(g),\xi^{-1}(gs)}(T^g f)) \\
&= \Phi^{-1}\Big(T^g f \otimes \Big(\sum_{s \in S} q^{\ell,1,2}_{\xi^{-1}(g),\xi^{-1}(gs)} \delta_{\xi^{-1}(gs)}\Big)\Big) \\
&= \Phi^{-1}(T^g f \otimes (Q^{\ell,1,2}(\delta_{\xi^{-1}(g)}))) = (\Phi^{-1} \circ (\mathrm{id}_{\mathfrak{h}_\mathcal{C}} \otimes Q^{\ell,1,2}) \circ \Phi)(f),
\end{aligned}$$

as required. □

4 Computation of an eigenspace

We now specialize to a particular selection of an eigenvalue of interest to us: the value 4. We will find (in Corollary 4.3 below) that we can describe the eigenspace $\ker(Q - 4 \cdot \mathrm{id})$ rather explicitly. The choice of 4 here is important: it originates in a particular quadratic equation that arises from a two-step linear recursion that will appear repeatedly below in the description of the associated eigenspaces. For this value we can obtain a proof of the non-existence of such eigenspaces for some of the restrictions $Q|_{\mathfrak{H}_\mathcal{C}}$, and an explicit construction of these eigenspaces for others.

We will find that (in much the same way as for the simple lamplighter group as described in Dicks and Schick [5]) we can arrange a 'pileup' of infinitely many eigenspaces corresponding to this eigenvalue, each of them admitting a relatively simple description, and it is this that will ultimately give us the control over von Neumann dimensions required for Theorem 1.1.

Lemma 4.1 *The value 4 is not an eigenvalue of $Q|_{\mathfrak{H}_\mathcal{C}}$ for any $\mathcal{C} \in \Omega_{1,2}/\sim$ or $\mathcal{C} \in \Omega_{2,2}/\sim$.*

Proof We give the proof for $\mathcal{C} \in \Omega_{1,2}/\sim$, the other case being exactly similar. By Proposition 3.12 it will suffice to show that

$$\text{for } \ell \geq 5,\ \mathbf{x} \in \ell^2(V^{\ell,1,2}), \qquad Q^{\ell,1,2}\mathbf{x} = 4\mathbf{x} \quad \Rightarrow \quad \mathbf{x} = \mathbf{0}.$$

To this end, enumerate the central length-$(\ell-1)$ path of $V^{\ell,1,2}$ as $V_0 = \{v_1, v_2, \ldots, v_\ell\}$, and observe that each pair of neighbours in this path is joined by an edge of weight 2, that each vertex in this path is joined to exactly two non-members of this path by edges of weight 2, and that one of the end-points is additionally joined to a non-member of this path by an edge of weight $\frac{101}{100}$. These are all the positive-weight edges in the graph. For each $i \in \{1, 2, \ldots, \ell-1\}$ let $v_{i,j}$ for $j = 1, 2$ be the two neighbours of v_i in $V^{\ell,1,2} \setminus V_0$, and also let $v_{\ell,1}$ and $v_{\ell,2}$ be the two outer neighbours of v_ℓ joined to it by a weight of 2, and v'_ℓ the neighbour of v_ℓ joined to it by the weight $\frac{101}{100}$. Finally let

$$\omega := \frac{1+\sqrt{-3}}{2},$$

so that $\{1, \omega, \omega^2, -1, \bar{\omega}^2, \bar{\omega}\}$ are the sixth roots of unity.

We can evaluate the equation $Q^{\ell,1,2}\mathbf{x} = 4\mathbf{x}$ at v_i for $2 \leq i \leq \ell-1$ and also at $v_{i,j}$ for such i, and find that

$$\begin{aligned}
\text{at } v_{i,j} \quad : \qquad & 4\mathbf{x}(v_{i,j}) = 2\mathbf{x}(v_i) \quad \Rightarrow \quad \mathbf{x}(v_{i,j}) = \frac{1}{2}\mathbf{x}(v_i) \\
\text{at } v_i \quad : \qquad & 4\mathbf{x}(v_i) = 2\mathbf{x}(v_{i-1}) + 2\mathbf{x}(v_{i+1}) + 2\mathbf{x}(v_{i,1}) + 2\mathbf{x}(v_{i,2}) \\
& \quad \Rightarrow \quad \mathbf{x}(v_i) = \mathbf{x}(v_{i-1}) + \mathbf{x}(v_{i+1}),
\end{aligned}$$

and so re-arranging we obtain

$$\mathbf{x}(v_{i+1}) = \mathbf{x}(v_i) - \mathbf{x}(v_{i-1}) \qquad \forall i = 2, 2, \ldots, \ell-1,$$

and hence by solving this quadratic recursion that there are $a, b \in \mathbb{C}$ such that $\mathbf{x}(v_i) = a\omega^i + b\bar{\omega}^i$; and we also obtain similarly that $\mathbf{x}(v_{i,j}) = \frac{1}{2}\mathbf{x}(v_i)$ for all $i \in \{1, 2, \ldots, \ell\}$ and $j = 1, 2$.

Next, evaluating at v'_ℓ gives that $\mathbf{x}(v'_\ell) = \frac{101}{400}\mathbf{x}(v_\ell)$, and now evaluating at v_ℓ gives

$$\begin{aligned}
4\mathbf{x}(v_\ell) &= 2\mathbf{x}(v_{\ell-1}) + 2(\mathbf{x}(v_{\ell,1}) + \mathbf{x}(v_{\ell,2})) + \frac{101^2}{4\cdot 100^2}\mathbf{x}(v_\ell) \\
&= 2\mathbf{x}(v_{\ell-1}) + 2\mathbf{x}(v_\ell) + \frac{101^2}{4\cdot 100^2}\mathbf{x}(v_\ell) \\
\Rightarrow & \qquad \Big(1 - \frac{101^2}{8\cdot 100^2}\Big)(a\omega^\ell + b\bar{\omega}^\ell) = a\omega^{\ell-1} + b\bar{\omega}^{\ell-1}.
\end{aligned}$$

It follows that either $\mathbf{x} = \mathbf{0}$ or at least one of a, b is non-zero. Let us suppose it is b and derive a contradiction, the case $a \neq 0$ being similar. In this case the above conclusion can be re-arranged to give

$$\Big(\Big(1 - \frac{101^2}{8\cdot 100^2}\Big)\omega^\ell - \omega^{\ell-1}\Big)\frac{a}{b} = \bar{\omega}^{\ell-1} - \Big(1 - \frac{101^2}{8\cdot 100^2}\Big)\bar{\omega}^\ell,$$

and now evaluating the eigenvector equation at v_1 (the only vertex where we have not yet checked it) gives similarly

$$a\omega + b\bar{\omega} = \mathbf{x}(v_1) = \mathbf{x}(v_2) = a\omega^2 + b\bar{\omega}^2 \qquad \Rightarrow \qquad (\omega - \omega^2)\frac{a}{b} = \bar{\omega}^2 - \bar{\omega}.$$

It can now be verified directly that no value $\frac{a}{b}$ can simultaneously satisfy both of the above equations (bearing in mind that the sequence $(\omega^\ell)_{\ell \geq 1}$ takes only six values). This gives the desired contradiction, and so completes the proof. $\square$

Lemma 4.2 *The value* 4 *is an eigenvalue of* $Q^{\ell,1,1}$ *with multiplicity* 1 *whenever* $\ell \equiv -1 \mod 6$*, and hence also of* $Q|_{\mathfrak{H}_\mathcal{C}}$ *for any* $\mathcal{C} \in \Omega_{1,1}/\sim$ *such that* $(P,\psi) \in \mathcal{C}$ *has* $|P| \equiv -1 \mod 6$ *and* $\mathfrak{h}_\mathcal{C} \neq \{0\}$.

Proof If $\mathfrak{h}_\mathcal{C} \neq \{0\}$ then the conclusion for $Q|_{\mathfrak{H}_\mathcal{C}}$ follows directly from that for $Q^{\ell,1,1}$ using Proposition 3.12, so we focus on the latter. We explicitly exhibit a suitable eigenvector, and then the argument of the preceding lemma shows that it is the only one up to scalar multiples. Let $V_0 = \{v_1, v_2, \ldots, v_\ell\}$, $v_{i,j}$ for $j = 1,2$ and $\omega \in \mathbb{C}$ be as in the preceding lemma, and now define $\mathbf{x} \in \ell^2(V^{\ell,1,1})$ by

$$\mathbf{x}(v_i) := \omega^i + \bar{\omega}^{3+i} \qquad \text{and} \qquad \mathbf{x}(v_{i,j}) := \frac{1}{2}(\omega^i + \bar{\omega}^{3+i})$$

for $v_i, v_{i,j} \in V^{\ell,1,1}$.

It is now a simple check that $Q^{\ell,1,1}\mathbf{x} = 4\mathbf{x}$:

- At v_i for $2 \leq i \leq \ell - 1$ we have

$$\begin{aligned}(Q^{\ell,1,1}\mathbf{x})(v_i) &= 2\mathbf{x}(v_{i-1}) + 2\mathbf{x}(v_{i+1}) + 2\mathbf{x}(v_{i,1}) + 2\mathbf{x}(v_{i,2}) \\ &= 2(\omega^{i-1} + \bar{\omega}^{3+i-1} + \omega^{i+1} + \bar{\omega}^{3+i+1} + \omega^i + \bar{\omega}^{3+i}) \\ &= 2\big((\omega^i + \bar{\omega}^{3+i}) + (\omega^i + \bar{\omega}^{3+i})\big) \\ &= 4(\omega^i + \bar{\omega}^{3+i}) = 4\mathbf{x}(v_i),\end{aligned}$$

  since $\omega + \bar{\omega} = 1$.

- At v_1 we have $\mathbf{x}(v_1) = \omega + \bar{\omega}^4 = \omega + \omega^2$, and

$$\begin{aligned}(Q^{\ell,1,1}\mathbf{x})(v_1) &= 2\mathbf{x}(v_2) + 2\mathbf{x}(v_{1,1}) + 2\mathbf{x}(v_{2,2}) \\ &= 2\Big(\omega^2 + \bar{\omega}^5 + 2 \cdot \frac{1}{2}(\omega + \omega^2)\Big) = 4(\omega + \omega^2) = 4\mathbf{x}(v_1),\end{aligned}$$

using now that $\bar{\omega}^5 = \omega$, and similarly since $\ell \equiv -1 \mod 6$ we have

$$\begin{aligned}(Q^{\ell,1,1}\mathbf{x})(v_\ell) &= 2\mathbf{x}(v_{\ell-1}) + 2\mathbf{x}(v_{\ell,1}) + 2\mathbf{x}(v_{\ell,2}) \\ &= 2\Big(\omega^{\ell-1} + \bar{\omega}^{\ell+2} + 2\cdot\frac{1}{2}(\omega^{\ell} + \bar{\omega}^{3+\ell})\Big) = 4(\omega^{-1} + \bar{\omega}^2) = 4\mathbf{x}(v_\ell)\end{aligned}$$

(notice, however, that more general linear combinations of ω^i and $\bar{\omega}^i$ would not work here);

- Finally, at a leaf $v_{i,j}$ we need $2\mathbf{x}(v_i) = 4\mathbf{x}(v_{i,j})$, and this is obvious. $\square$

Combining the above calculations now gives the following.

Corollary 4.3 *With* $\Gamma := (\mathbb{Z}_2^{\oplus \mathbf{F}_2}/V) \rtimes \mathbf{F}_2$, *under the assumption that* $\mathfrak{H}_{1,\infty} = \mathfrak{H}_{2,\infty} = \mathfrak{H}_{\infty,\infty} = \{0\}$ *we have*

$$\ker(Q - 4\cdot \mathrm{id}_{L^2(m_{V^\perp}\otimes \#_{\mathbf{F}_2})}) = \bigoplus_{i\geq 1} \ker(Q|_{\mathfrak{H}_{\mathcal{C}_i}} - 4\cdot\mathrm{id}_{\mathfrak{H}_{\mathcal{C}_i}})$$

for some infinite sequence $\mathcal{C}_1, \mathcal{C}_2, \dots$, *in* $\Omega_{1,1}/\sim$, *and hence*

$$\begin{aligned}\dim_{L\Gamma}\ker(Q - 4\cdot\mathrm{id}_{L_2(m_{V^\perp}\otimes\#_{\mathbf{F}_2})}) &= \sum_{i\geq 1}\dim_{L\Gamma}\ker(Q|_{\mathfrak{H}_{\mathcal{C}_i}} - 4\cdot\mathrm{id}_{\mathfrak{H}_{\mathcal{C}_i}}) \\ &= \sum_{i\geq 1}\dim_{L\Gamma}\,\mathfrak{h}_{\mathcal{C}_i}.\end{aligned}$$

Proof This all follows directly from the preceding lemmas upon noting that since the value 4 has multiplicity 1 as an eigenvalue of $Q^{\ell,1,1}$ we have

$$\begin{aligned}\dim_{L\Gamma}\ker(Q|_{\mathfrak{H}_{\mathcal{C}_i}} - 4\cdot\mathrm{id}_{\mathfrak{H}_{\mathcal{C}_i}}) &= \dim_{L\Gamma}\ker(\mathrm{id}_{\mathfrak{h}_{\mathcal{C}_i}} \otimes (Q^{\ell,1,1} - 4\cdot\mathrm{id}_{\ell^2(V^{\ell,1,1})}) \\ &= \dim_{L\Gamma}\,\mathfrak{h}_{\mathcal{C}_i}.\end{aligned}$$

$\square$

Definition 4.4 *We will refer to those* $\mathcal{C} \in \Omega_{1,1}/\sim$ *that contribute nontrivially to the above sum expression for* $\dim_{L\Gamma}\ker(Q - 4\cdot\mathrm{id}_{L_2(m_{V^\perp}\otimes\#_{\mathbf{F}_2})})$, *or also any* (P,ψ) *that lies in such a* $\mathcal{C}$, *as* ***active****.*

5 Estimates on von Neumann dimensions

So far our results have been independent of the particular choice of the subspace V, and in particular of the Haar measure $m_{V^\perp}$, even though it has already been mentioned in the notation a number of times. That choice will now become important, as we seek to show how certain possible choices of V give different possible values for the von Neumann dimensions of the subspaces in Corollary 4.3.

The calculation of these dimensions will rest on the following lemma.

Lemma 5.1 *Suppose that $V \leq \mathbb{Z}_2^{\mathbf{F}_2}$ is a subgroup, $A \subset \mathbf{F}_2$ is a finite subset and for $\psi : A \to \mathbb{Z}_2$ let*

$$C(\phi) := \{\boldsymbol{\chi} \in \mathbb{Z}_2^{\mathbf{F}_2} :\ \boldsymbol{\chi}|_A = \phi\}.$$

Then

$$m_{V^\perp}(C(\phi)) = \begin{cases} \frac{1}{|\{\phi' \in \mathbb{Z}_2^A :\ C(\phi') \cap V^\perp \neq \emptyset\}|} & \text{if } C(\phi) \cap V^\perp \neq \emptyset \\ 0 & \text{else} \end{cases}$$

(that is, the measure $m_{V^\perp}$ is shared equally among those cylinder sets $C(\phi)$ that intersect $V^\perp$ nontrivially).

Proof Clearly $m_{V^\perp}(C(\phi)) = 0$ if $C(\phi) \cap V^\perp = \emptyset$, so it suffices to prove that every $C(\phi)$ for which $C(\phi) \cap V^\perp \neq \emptyset$ has equal measure under $m_{V^\perp}$. If $C(\phi_1), C(\phi_2)$ are two such, then we can pick some $\boldsymbol{\chi}_i \in C(\phi_i) \cap V^\perp$ for $i = 1, 2$, and now inside the group $\mathbb{Z}_2^{\mathbf{F}_2}$ the translation by $\boldsymbol{\chi}_2 - \boldsymbol{\chi}_1$ is $m_{V^\perp}$-preserving and sends $C(\phi_1) \cap V^\perp$ to $C(\phi_2) \cap V^\perp$, so this completes the proof. $\square$

We now turn to the steps needed in our construction of the subgroups V_I. Our first step is to pick a strictly increasing sequence $(l(n))_{n \geq 1}$ in $\mathbb{N}$ (where we adopt the convention $0 \notin \mathbb{N}$).

Lemma 5.2 *The elements $t_i := s_2^{l(i)} s_1 s_2^{-l(i)}$, $i \geq 1$, are free in $\mathbf{F}_2$, and so generate a homomorphic embedding $\mathbf{F}_\infty \hookrightarrow \mathbf{F}_2$.*

Proof Suppose that

$$t_{i_1}^{k_1} t_{i_2}^{k_2} \cdots t_{i_m}^{k_m} = e$$

for some sequences $i_1, i_2, \dots, i_m \in \{0, 1, \dots, n\}$ and $k_1, k_2, \dots, k_m \in \mathbb{Z} \setminus \{0\}$. Then since $t_i^k = s_2^{l(i)} s_1^k s_2^{-l(i)}$ for all $k \in \mathbb{Z}$, we may reduce this evaluation to

$$s_2^{l(i_1)} s_1^{k_1} s_2^{l(i_2)-l(i_1)} s_1^{k_2} s_2^{l(i_3)-l(i_2)} \cdots s_2^{k_m} s_1^{-l(i_m)} = e,$$

and it is now clear that this is possible only if $i_1 = i_2 = \ldots = i_m$ and $k_1 + k_2 + \cdots + k_m = 0$, hence only if the original word was trivial. $\square$

Lemma 5.3 *For any $h \in \langle t_n : n \geq 1\rangle \setminus \{e\}$ the path in $\mathrm{Cay}(\mathbf{F}_2, S)$ joining e to h passes through $s_2^{\pm 1}$ and not through $s_1^{\pm 1}$.*

Proof If

$$h = t_{i_1}^{k_1} t_{i_2}^{k_2} \cdots t_{i_m}^{k_m}$$

for some $i_1, i_2, \ldots, i_m \in \{1, 2, \ldots, n\}$ with consecutive values distinct and some $k_1, k_2, \ldots, k_m \in \mathbb{Z} \setminus \{0\}$, then as before we can write this out as

$$s_2^{l(i_1)} s_1^{k_1} s_2^{l(i_2)-l(i_1)} s_1^{k_2} s_2^{l(i_3)-l(i_2)} \cdots s_2^{k_m} s_1^{-l(i_m)},$$

and this is now the reduced word form of h. Since the path in question is just the sequence of initial segments of this word, we can see that the first step must be $s_2^{\pm 1}$, as required. $\square$

Now for $I \subseteq \mathbb{N}$ we define

$$V_I := \mathrm{span}_{\mathbb{Z}_2}\Big\{ \sum_{i=-10}^{10} (\delta_{gs_1^i} - \delta_{gt_n s_1^i}) : \ g \in \mathbf{F}_2, \ n \in I \Big\},$$

so that

$$V_I^{\perp} := \Big\{ \boldsymbol{\chi} \in \mathbb{Z}_2^{\mathbf{F}_2} : \ \sum_{i=-10}^{10} \boldsymbol{\chi}(gs_1^i) = \sum_{i=-10}^{10} \boldsymbol{\chi}(gt_n s_1^i) \ \forall g \in \mathbf{F}_2, \, n \in I \Big\}.$$

Let us also write $\Gamma_I := (\mathbb{Z}_2^{\oplus \mathbf{F}_2}/V_I) \rtimes \mathbf{F}_2$ and let Q_I be the operator in $\mathbb{Q}\Gamma_I$ defined as in Section 3. Finally, let

$$\Lambda_I := \langle t_n : \ n \in I \rangle \leq \mathbf{F}_2,$$

and for any subset $A \subseteq \mathbf{F}_2$ let

$$A/\Lambda_I := \{ A \cap g\Lambda_I : \ g \in A \},$$

the partition of A induced by the partition of $\mathbf{F}_2$ into left-cosets of Λ_I.

Lemma 5.4 *Let $\mathcal{C}_1, \mathcal{C}_2, \ldots \in \Omega_{1,1}/\sim$ be the active equivalence classes, and for each $i \in \mathbb{N}$ let $(P_i, \psi_i) \in \mathcal{C}_i$ be a representative for which $\mathfrak{h}_{\mathcal{C}_i} = \mathfrak{H}_{P_i,\psi_i}$. Then for any $I \subseteq \mathbb{N}$ we have*

$$\begin{aligned} \dim_{L\Gamma_I} \ker(Q_I - 4) &= \sum_{i \geq 1} m_{V_I^\perp}(C_{P_i,\psi_i}) & (2) \\ &= \sum_{i \geq 1} \frac{1_{\{C_{P_i,\psi_i} \cap V_I^\perp \neq \emptyset\}}}{|\{\phi \in \mathbb{Z}_2^{B(P_i,10)} :\ C(\phi) \cap V_I^\perp \neq \emptyset\}|}. & (3) \end{aligned}$$

Proof This follows simply from evaluating the individual terms in the right-hand side of Corollary 4.3 and observing directly from the formula for the trace on $L^\infty(m_{\widehat{U}}) \rtimes \mathbf{F}_2$ that

$$\dim_{L\Gamma_I} \mathfrak{h}_{\mathcal{C}_i} = \mathrm{tr}_{L\Gamma_I} M_{C_{P_i,\psi_i}} = m_{V^\perp}(C_{P_i,\psi_i}).$$

The second line now follows from Lemma 5.1. $\square$

Next we need a criterion for deciding whether $C(\phi) \cap V_I^\perp = \emptyset$.

Lemma 5.5 (Extensibility lemma) *If $A \subseteq \mathbf{F}_2$ is connected in the graph $\mathrm{Cay}(\mathbf{F}_2, S)$ and $\phi : A \to \mathbb{Z}_2$ is such that*

$$\sum_{i=-10}^{10} \phi(gs_1^i) = \sum_{i=-10}^{10} \phi(ghs_1^i) \qquad (4)$$

whenever $g \in \mathbf{F}_2$ and $h \in \Lambda_I$ are such that gs_1^{-10}, gs_1^{-9}, $\ldots$, gs_1^{10}, ghs_1^{-10}, ghs_1^{-9}, $\ldots$ and ghs_1^{10} all lie in A, then ϕ admits an extension $\boldsymbol{\chi} \in C(\phi) \cap V_I^\perp$.

Remark Both the connectedness assumption on A and the fact that the removal of any vertex from $\mathrm{Cay}(\mathbf{F}_2, S)$ disconnects this graph are important for this proof. $\lhd$

Proof Given $\boldsymbol{\chi} \in C(\phi)$, it is a member of $V_I^\perp$ if and only if for some (and hence any) upwards directed family of subsets $B \subseteq \mathbf{F}_2$ that covers all of $\mathbf{F}_2$ we have that for each B in the family the condition (4) holds whenever gs_1^{-10}, gs_1^{-9}, $\ldots$, gs_1^{10}, ghs_1^{-10}, ghs_1^{-9}, $\ldots$ and ghs_1^{10} all lie in B. Now let

$$A_0 = A \subset A_1 \subset A_2 \subset \ldots \subset \mathbf{F}_2$$

be an exhaustion of $\mathbf{F}_2$ in which each A_{n+1} is obtained from A_n by the inclusion of a single new point from $B(A_n, 1) \setminus A_n$ (clearly such an exhaustion exists). If we show how to construct recursively a sequence of functions $\boldsymbol{\chi}_n : A_n \to \mathbb{Z}_2$ for $n \geq 0$ such that

- $\boldsymbol{\chi}_0 := \phi$,
- $\boldsymbol{\chi}_{n+1}|_{A_n} = \boldsymbol{\chi}_n$ for all $n \geq 0$ and
- condition (4) is satisfied by $\boldsymbol{\chi}_n$ whenever $gs_1^{-10}, \ldots, gs_1^{10}, ghs_1^{-10}, \ldots$ and ghs_1^{10} all lie in A_n,

then it follows that $(\cup_{n\geq 1}\boldsymbol{\chi}_n) \in C(\phi) \cap V^{\perp}$ is the desired point.

Moreover, having set $\boldsymbol{\chi}_0 := \phi$, it suffices to give the construction for $\boldsymbol{\chi}_1$, since then simply repeating this construction with A_n in place of A at every step completes the proof.

To this end, suppose $A_1 = A \cup \{g_1\}$, let us write $\mathcal{E}(A)$ for the set of all equations of the form (4) for which $gs_1^{-10}, \ldots, gs_1^{10}, ghs_1^{-10}, \ldots$ and ghs_1^{10} all lie in A_1, and let us partition this as

$$\mathcal{E}(A) = \mathcal{E}_0(A) \cup \mathcal{E}_1(A),$$

where $\mathcal{E}_0(A)$ contains those equations that do not involve the value of $\boldsymbol{\chi}_1(g)$ and $\mathcal{E}_1(A)$ contains those that do. All members of $\mathcal{E}_0(A)$ are satisfied by our assumptions on ϕ, whereas each member of $\mathcal{E}_1(A)$ prescribes a value for $\boldsymbol{\chi}_1(g)$ in terms of values of ϕ. If $\mathcal{E}_1(A) = \emptyset$ then we may adopt either possible value for $\boldsymbol{\chi}_1(g)$, so it suffices to show that if $\mathcal{E}_1(A) \neq \emptyset$ then all the resulting prescriptions agree. To see this, observe that any two of these equations from $\mathcal{E}_1(A)$ must take the form

$$\boldsymbol{\chi}(g_1) = -\sum_{-10\leq i\leq 10,\, gs_1^i \neq g_1} \phi(gs_1^i) + \sum_{i=-10}^{10} \phi(gh_i s_1^i)$$

for some $g \in A$ and $h_1, h_2 \in \Lambda_I$. However, if

$$g_1 \in \{gs_1^{-10}, gs_1^{-9}, \ldots, gs_1^{10}\} \subset A_1,$$

then g_1 must be one of the end-points $gs_1^{\pm 10}$, for otherwise $g_1 \notin A$ would separate A into the two connected components containing these two end-points, contrary to our assumption that A is connected. Moreover, if $g' \in A$ is another point such that

$$g \in \{g's_1^{-10}, g's_1^{-9}, \ldots, g's_1^{10}\} \subset A_1,$$

then we must have $g = g'$, for if alternatively $gs_1^{-10} = g_1 = g's_1^{10}$ then g_1 disconnects the components of A that contain g and g'. Hence we may assume without loss of generality that all of the above equations from the collection $\mathcal{E}_1(A)$ have $gs_1^{10} = g_1$. However, since $h_1^{-1}h_2 \in \Lambda_I$, we now see that the equation

$$\sum_{i=-10}^{10} \phi((gh_1)s_1^i) = \sum_{i=-10}^{10} \phi((gh_1)h_1^{-1}h_2 s_1^i) = \sum_{i=-10}^{10} \phi(gh_2 s_1^i)$$

is a member of $\mathcal{E}_0(A)$ and so is satisfied by assumption; this implies that the right-hand-sides above are equal for the equations in $\mathcal{E}_1(A)$ corresponding to h_1 and to h_2, and hence prescribe a consistent value for $\boldsymbol{\chi}_1(gs_1^{10})$, as required. $\square$

Corollary 5.6 *If $P \subset \mathbf{F}_2$ is a path and $\phi : B(P, 10) \to \mathbb{Z}_2$ then $C(\phi) \cap V_I^\perp \neq \emptyset$ if and only the function*

$$P \to \mathbb{Z}_2 : g \mapsto \sum_{i=-10}^{10} \phi(gs_1^i)$$

is constant on the cells of P/Λ_I.

Proof The necessity is obvious, and the sufficiency follows from the previous lemma and the fact that

$$\{gs_1^{-10}, gs_1^{-9}, \dots, gs_1^{10}\} \subset B(P, 10) \qquad \Rightarrow \qquad g \in P.$$

This follows from the connectedness of P, because there must be some $g_1, g_2 \in P$ that lie within distance 10 of gs_1^{-10} and gs_1^{10} respectively, and were g not itself a member of P then these two other members of P would occupy distinct connected components, giving a contradiction. $\square$

Corollary 5.7 *If $P \subset \mathbf{F}_2$ is a path with no small horizontal doglegs and $\phi : B(P, 10) \to \mathbb{Z}_2$ takes the value 0 inside P and 1 on $B(P, 10) \setminus P$ then $C(\phi) \cap V_I^\perp \neq \emptyset$.*

Remark It is in this proof that we will finally see the purpose of the assumption of no small horizontal doglegs. $\triangleleft$

Proof By the previous corollary this depends only on the constancy of the values

$$\sum_{i=-10}^{10} \phi(gs_1^i),\ g \in P$$

on each cell of P/Λ_I. For ϕ as described this value is just

$$\begin{aligned} &|\{gs_1^{-10}, gs_1^{-9}, \dots, gs_1^{10}\} \cap (B(P, 10) \setminus P)| \mod 2 \\ &\qquad\qquad\qquad \equiv |\{gs_1^{-10}, gs_1^{-9}, \dots, gs_1^{10}\} \setminus P| \mod 2. \end{aligned}$$

If g lies in a singleton cell of P/Λ_I then there is nothing to check. On the other hand, if $g, gh \in P$ for some $h \in \Lambda_I \setminus \{e\}$, then by applying Lemma 5.3 to the

segment of P joining g and gh it follows that we must have $gs_2^\eta \in P$ for some $\eta = \pm 1$ and ghs_2^η for some $\eta = \pm 1$. From this it follows that $gs_1^{\pm 1}$ cannot both lie in P and that $ghs_1^{\pm 1}$ cannot both lie in P. Hence the intersection

$$P \cap \{gs_1^{-10}, gs_1^{-9}, \ldots, gs_1^{10}\}$$

is either just $\{g\}$, in which case

$$|\{gs_1^{-10}, gs_1^{-9}, \ldots, gs_1^{10}\} \setminus P| = 20 \equiv 0 \mod 2;$$

or else it also contains some point gs_1^a with $a \neq 0$, so that by the assumption of no small horizontal doglegs it must in fact contain exactly one of the whole branches

$$\{gs_1^{-10}, gs_1^{-9}, \ldots, g\} \qquad \text{or} \qquad \{g, gs_1, \ldots, gs_1^{10}\},$$

in which case

$$|\{gs_1^{-10}, gs_1^{-9}, \ldots, gs_1^{10}\} \setminus P| = 10 \equiv 0 \mod 2.$$

Thus the value in question is always $0 \in \mathbb{Z}_2$ for those g lying in a nonsingleton cell of P/Λ_I, and so we have proved the necessary constancy on these cells. $\square$

We will now use the preceding lemmas and corollaries to two distinct ends. We first show that we must have

$$m_{V_I^\perp}(C_{1,\infty}) = m_{V_I^\perp}(C_{2,\infty}) = m_{V_I^\perp}(C_{\infty,\infty}) = 0 \qquad \forall I \subseteq \mathbb{N}.$$

Combined with Lemma 4.1, this justifies restricting our attention to $Q|_{\mathfrak{H}_\mathcal{C}}$ for $\mathcal{C} \in \Omega_{1,1}/\sim$ when calculating $\ker(Q - 4 \cdot \mathrm{id})$. We will then give that calculation, and use it to deduce the monotonicity needed for Theorem 1.1.

Proposition 5.8 *For any $I \subseteq \mathbb{N}$ we have*

$$m_{V_I^\perp}(C_{1,\infty}) = m_{V_I^\perp}(C_{2,\infty}) = m_{V_I^\perp}(C_{\infty,\infty}) = 0.$$

Proof If

$$\boldsymbol{\chi} \in C_{1,\infty} \cup C_{2,\infty} \cup C_{\infty,\infty}$$

then, in particular, there is some $g \in S \cup \{e\}$ and some singly-infinite path $P = \{g_1, g_2, \ldots\} \subseteq \boldsymbol{\chi}^{-1}\{0\}$ starting from $g_1 \in \partial\{g\}$, and such that for any $h \in B(P, 10) \setminus P$ whose connection to P does not pass through g we have $\boldsymbol{\chi}(h) = 1$. Now given any $g_0 \in \mathbf{F}_2$ and $g_1 \in g_0 S$, let $K \subset \mathbf{F}_2$ be the quadrant of points h

that are not disconnected from g_1 by g_0. Since $B(\{e\},2)$ is finite, it will suffice to prove that for any fixed such g_0 and g_1 we have

$$m_{V^\perp}\big\{\boldsymbol{\chi}:\ \boldsymbol{\chi}^{-1}\{0\} \text{ connects } g_1 \text{ to } \infty \text{ inside } K \text{ along some path } P \\ \text{and } \boldsymbol{\chi}|_{(B(P,10)\cap K)\setminus P} \equiv 1\big\} = 0.$$

This, in turn, will follow if we show that $m_{V^\perp}(D_N) \to 0$ as $N \to \infty$ where

$$D_N := \big\{\boldsymbol{\chi}:\ \boldsymbol{\chi}^{-1}\{0\} \text{ connects } g_1 \text{ to } \partial B(g_1,N)\cap K \text{ along some path } P \\ \text{and } \boldsymbol{\chi}|_{(B(P,10)\cap B(g_1,N)\cap K)\setminus P} \equiv 1\big\}.$$

Now for each path P that connects g_1 to $\partial B(g_1,N)$ inside K we let

$$D_{N,P} := \big\{\boldsymbol{\chi}:\ P \subseteq \boldsymbol{\chi}^{-1}\{0\} \text{ and } \boldsymbol{\chi}|_{(B(P,10)\cap B(g_1,N)\cap K)\setminus P} \equiv 1\big\},$$

and now we have $D_N = \bigcup_P D_{N,P}$. Finally, on the one hand we know that there are at most 3^N such paths P, and on the other we know that P/Λ_I has size at most $|P| = N$ for any P and any $I \subseteq \mathbb{N}$, and hence Lemma 5.1 and Corollary 5.6 give

$$m_{V^\perp}(D_{N,P}) \leq \frac{1}{2^{|B(P,10)\cap B(g_1,N)\cap K| - |P/\Lambda_I|}} \leq \frac{2^N}{2^{(2\cdot 3^9)(N-10)}}.$$

Combining these estimates gives

$$m_{V^\perp}(D_N) \leq 3^N \cdot 2^N \cdot 2^{-(2\cdot 3^9)(N-10)} \to 0 \qquad \text{as } N \to \infty,$$

as required. $\square$

Remark In fact, one can use a different argument to prove that

$$\nu(C_{1,\infty}) = \nu(C_{2,\infty}) = 0$$

for *any* left-translation-invariant Borel probability measure ν on $\mathbb{Z}_2^{\mathbf{F}_2}$. Any element $\boldsymbol{\chi} \in C_{1,\infty}$ gives rise to a semi-infinite path $P(\boldsymbol{\chi}) \subseteq \mathbf{F}_2$ that passes within distance 1 of e by selecting the connected component of $\boldsymbol{\chi}^{-1}\{0\}$ closest to e. Based on this we may classify members of $C_{1,\infty}$ according to the position of e relative to this path, where we record that position using some labeling of the vertices in $B(P(\boldsymbol{\chi}),1)$ by the vertices of the infinite tree that has a semi-infinite central path and two extra leaves adjacent to every point of that path (where just a little care is needed so that the labeling is a Borel function of $\boldsymbol{\chi}$). This now gives a Borel partition of $C_{1,\infty}$ into countably many cells indexed by the vertices of this infinite tree, and these cells are now easily seen to be related one to another by members of the full group of measure-preserving transformations of $(\mathbb{Z}_2^{\mathbf{F}_2},\nu)$

generated by the coordinate-translation action of $\mathbf{F}_2$. It follows that these countably many cells must all have the same measure, and hence that that measure is zero. However, this argument breaks down for doubly-infinite paths because we have no end-point of the path to use a reference so as to define our labeling of the members of $B(P,1)$ in a Borel way; and, indeed, it is not hard to find some examples of translation invariant probability ν under which *almost every* $\boldsymbol{\chi} \in \mathbb{Z}_2^{\mathbf{F}_2}$ is such that $\boldsymbol{\chi}^{-1}\{0\}$ is a union of disconnected doubly-infinite paths, one of which paths often passes close to e. In view of this it seems easier to treat all three cases together using the more analytic argument above. $\square$

Corollary 5.9 *For any finite path $P \subset \mathbf{F}_2$ we have*

$$|\{\phi \in \mathbb{Z}_2^{B(P,10)} :\ C(\phi) \cap V_I^\perp \neq \emptyset\}| = 2^{|B(P,10)|-|P/\Lambda_I|},$$

and so

$$\dim_{L\Gamma_I} \ker(Q_I - 4) = \sum_{i\geq 1} 2^{-(|B(P_i,10)|-|P_i|+|P_i/\Lambda_I|)}.$$

Proof From Corollary 5.6 and the standard relation $|F| \cdot |F^\perp| \equiv 2^N$ for subgroups $F \leq \mathbb{Z}_2^N$ we can identify

$$\{\phi \in \mathbb{Z}_2^{B(P_i,10)} :\ C(\phi) \cap V_I^\perp \neq \emptyset\}$$

with the subgroup of those $\phi \in \mathbb{Z}_2^{B(P_i,10)}$ that annihilates all the vectors of the form

$$\sum_{i=-10}^{10} \delta_{gs_1^i} - \sum_{i=-10}^{10} \delta_{ghs_1^i}$$

such that $h \in \Lambda_I$ and g and hg both lie in P_i. Clearly each cell $C \in P_i/\Lambda_I$ gives rise to a subspace of $\mathbb{Z}_2^{B(P_i,10)}$ of dimension $|C|-1$ spanned by these differences with $g, hg \in C$, and so the total dimension of the resulting subspace is

$$\sum_{C\in P_i/\Lambda_I} (|C|-1) = |P_i| - |P_i/\Lambda_I|.$$

This gives the dimension of

$$\{\phi \in \mathbb{Z}_2^{B(P_i,10)} :\ C(\phi) \cap V_I^\perp \neq \emptyset\}$$

as $|B(P_i,10)| - |P_i| + |P_i/\Lambda_I|$, and so both the desired conclusions now follow from Lemma 5.4. $\square$

Proof of Theorem 1.1 We will show that the conclusion holds for the parameterized family of subgroups V_I and the operators $Q_I - 4$ in place of Q_I provided the sequence of lengths $l(n)$, $n \geq 1$, appearing in the definition of t_n grows sufficiently fast.

Letting

$$\varphi(I) := \dim_{L\Gamma_I} \ker(Q_I - 4) \qquad \text{for } I \subseteq \mathbb{N},$$

we must prove that

$$I <_{\mathrm{lex}} J \qquad \Rightarrow \qquad \varphi(I) < \varphi(J)$$

provided that for each n the values $l(n')$ for $n' > n$ are sufficiently large relative to $l(1)$, $l(2)$, $\dots$, $l(n)$. More concretely, we will prove that if $l(1) < l(2) < \dots < l(n-1)$ and some auxiliary $L(n-1) > l(n-1)$ are such that the above implication holds whenever $I \cap [1, n-1] <_{\mathrm{lex}} J \cap [1, n-1]$ and for *any* tail sequence $L(n-1) \leq l(n) < l(n+1) < \dots$, then we can pick particular values of $l(n)$ and $L(n) > l(n)$ such that this same property holds with n in place of $n-1$. From this a simple recursion completes the proof.

Thus, suppose that $n \in \mathbb{N}$ is minimal such that $n \in J$ but $n \notin I$ (so by the definition of the lexicographic ordering we must have $I \cap [1, n-1] = J \cap [1, n-1]$). By Corollary 5.9 we have

$$\varphi(J) - \varphi(I) = \sum_{i \geq 1} \left(2^{-(|B(P_i,10)| - |P_i| + |P_i/\Lambda_J|)} - 2^{-(|B(P_i,10)| - |P_i| + |P_i/\Lambda_I|)}\right).$$

Clearly (from the freeness of the t_ns) there will be some paths P_i in the above list for which $P_i/\Lambda_{J\cap[1,n]}$ is a nontrivial coarsening of $P_i/\Lambda_{J\cap[1,n-1]} = P_i/\Lambda_{I\cap[1,n]}$ (that is, the left cosets of the larger subgroup $\Lambda_{J\cap[1,n]}$ intersect P_i in fewer, larger patches than those of the smaller subgroup $\Lambda_{J\cap[1,n-1]}$), and so the expression

$$2^{-(|B(P_i,10)| - |P_i| + |P_i/\Lambda_{J\cap[1,n]}|)} - 2^{-(|B(P_i,10)| - |P_i| + |P_i/\Lambda_{I\cap[1,n]}|)}$$

in the above sum will be strictly positive for each of these i. Let $E \subset \mathbb{N}$ be a finite subset of $i \in \mathbb{N}$ for which this is so, and such that

$$\sum_{i \in E} 2^{-(|B(P_i,10)| - |P_i| + |P_i/\Lambda_{J\cap[1,n]}|)} - 2^{-(|B(P_i,10)| - |P_i| + |P_i/\Lambda_{I\cap[1,n]}|)} =: \eta > 0.$$

Let $i_0 := \max\, E$. Next we observe, using only very crude estimates at every step,

that for any I and J and any $L \geq 1$ we have

$$\begin{aligned}
&\sum_{i\geq 1,\, |P_i|\geq L} \left(2^{-(|B(P_i,10)|-|P_i|+|P_i/\Lambda_J|)} - 2^{-(|B(P_i,10)|-|P_i|+|P_i/\Lambda_I|)}\right) \\
&\leq 2\cdot \sum_{i\geq 1,\, |P_i|\geq L} 2^{-(|B(P_i,10)|-|P_i|)} = 2\cdot \sum_{i\geq 1,\, |P_i|\geq L} 2^{|P_i|}2^{-|B(P_i,10)|} \\
&\leq 2\cdot \sum_{\substack{\text{all paths } P \text{ in } \mathrm{Cay}(\mathbf{F}_2,S) \\ \text{with } e\in B(P,1) \text{ and } |P|\geq L}} 2^{|P|}2^{-|B(P,10)|} \\
&\leq 2\cdot \sum_{\ell\geq L} \sum_{\substack{\text{all paths } P \text{ in } \mathrm{Cay}(\mathbf{F}_2,S) \\ \text{with } e\in B(P,1) \text{ and } |P|=\ell}} 2^{\ell}2^{-(2+2\cdot 3+\ldots+2\cdot 3^9)\ell} \\
&\leq 2\cdot \sum_{\ell\geq L} (3\ell+2)\cdot 3^{\ell}\cdot 2^{\ell}\cdot 2^{-(2+2\cdot 3+\ldots+2\cdot 3^9)\ell} \\
&< \infty.
\end{aligned}$$

Since for any finite L there can be only finitely many paths among P_1, P_2, $\ldots$ of length $< L$, it follows that

$$\sum_{i\geq i_1+1} \left(2^{-(|B(P_i,10)|-|P_i|+|P_i/\Lambda_J|)} - 2^{-(|B(P_i,10)|-|P_i|+|P_i/\Lambda_I|)}\right)$$

tends to 0 as $i_1 \to \infty$ uniformly fast in I and J, and so we may pick $i_1 > i_0$ such that

$$\sum_{i\geq i_1+1} \left(2^{-(|B(P_i,10)|-|P_i|+|P_i/\Lambda_J|)} - 2^{-(|B(P_i,10)|-|P_i|+|P_i/\Lambda_I|)}\right) < \eta/2$$

irrespective of the choice of $l(n+1)$, $l(n+2)$, $\ldots$. Therefore if we simply insist that these lengths $l(n')$ for $n' > n$ should be so large that

$$P_i/\Lambda_I = P_i/\Lambda_{I\cap[1,n]} \qquad \forall i \leq i_1,$$

we deduce that

$$\begin{aligned}
\varphi(J)-\varphi(I) &= \sum_{i=1}^{i_1}\big(2^{-(|B(P_i,10)|-|P_i|+|P_i/\Lambda_{J\cap[1,n]}|)}-2^{-(|B(P_i,10)|-|P_i|+|P_i/\Lambda_{I\cap[1,n]}|)}\big)\\
&\quad+\sum_{i\geq i_1+1}\big(2^{-(|B(P_i,10)|-|P_i|+|P_i/\Lambda_J|)}-2^{-(|B(P_i,10)|-|P_i|+|P_i/\Lambda_I|)}\big)\\
&\geq \sum_{i\in E}\big(2^{-(|B(P_i,10)|-|P_i|+|P_i/\Lambda_{J\cap[1,n]}|)}-2^{-(|B(P_i,10)|-|P_i|+|P_i/\Lambda_{I\cap[1,n]}|)}\big)\\
&\quad-\Big|\sum_{i\geq i_1+1}\big(2^{-(|B(P_i,10)|-|P_i|+|P_i/\Lambda_J|)}-2^{-(|B(P_i,10)|-|P_i|+|P_i/\Lambda_I|)}\big)\Big|\\
&\geq \eta-\eta/2=\eta/2>0,
\end{aligned}$$

as required. $\square$

Remarks **1.** We have presented the proof above so as to emphasize the flexibility in the choice of the sequence $(l(n))_{n\geq 1}\ \geq$, but easy estimates show, for example, that any doubly exponential sequence such as $l(n)=2^{2^n}$ will do.

2. Similar arguments also prove the continuity of the map ϕ for the product topology on $\mathcal{P}(\mathbb{N})$, but we do not need this. $\lhd$

6 Closing remarks

As they stand, the methods of this paper are too crude to touch what may be the most interesting special case of Atiyah's question: that for torsion-free groups. This asks for a torsion-free group and a rational group ring element $Q\in\mathbb{Q}\Gamma$ that has any nontrivial eigenspaces at all. In fact it is known that this is impossible for large classes of torsion-free groups (see, for example, Linnell's Theorem in [9] and Reich's thesis [12]), among which it implies such striking consequences as Kaplansky's conjecture that the group ring has no nontrivial zero-divisors, but the methods of the present paper do not offer any obvious hope of constructing a positive example.

Other natural classes of groups not obviously to be found among our family of examples might also be interesting to consider. In the case of amenable groups I think it very likely that a similar construction $(\mathbb{Z}_2^{\oplus\Lambda}/V)\rtimes\Lambda$ with a discrete amenable base group Λ should exist that admits an element of the rational group ring having kernel with irrational von Neumann dimension, but here the absence of the simple tree structure on $\mathrm{Cay}(\Lambda,S)$ will mandate some more delicate construction and estimates than we have used above. On the other hand, it is clear that

the basic counting trick that underlies Theorem 1.1 cannot be used to find finitely-presented examples, as there are only countably many of these, but it might be possible to make the estimates used above explicit enough to try to construct, for example, a finitely presented group admitting a rational group ring element with kernel dimension that is not explicitly evaluable, but can nevertheless be proved to be a Liouville number. The consideration of finitely presented groups, in particular, has some geometric interest since these are precisely the groups Γ for which proper cocompact free Γ-manifolds can be constructed as the universal covers $\widetilde{M}$ of compact manifolds M having $\pi_1(M) = \Gamma$.

In a slightly different direction, Theorem 1.1 and the remark following its proof give us a whole Cantor set of values in $[0, 1]$ as the possible von Neumann dimensions of our kernels. I suspect that a modification of the construction (possibly combined with some simple algebraic operations such as direct products) could be made to yield any real value in this interval, but this might also require a more careful look at the estimates involved.

It would also be interesting to know whether the semidirect product constructions we have used, which seem to offer a great deal of flexibility, could be brought to bear on the search for examples of new phenomena elsewhere in geometric group theory, such as groups not having the algebraic eigenvalue property ([6]). However, I have no more concrete suggestions in this direction to offer here.

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